Risk-Averse Optimization Formulation

- Recall, we want to solve this computationally expensive problem

$$\min_{z \in Z, t \in \mathbb{R}} \widehat{J}(t, z) \quad \text{where} \quad \widehat{J}(t, z) := \left\{ t + \frac{1}{N} \sum_{j=1}^{N} v\big(G(S(\xi; z), \xi) - t\big) \right\}$$

Inexact Trust Region Framework

Given an iterate $x_k = (t_k, z_k) \in X := \mathbb{R} \times Z$,

1. **Model Update: Choose s new model** $m_k \approx \widehat{J}(x_k + s)$.
2. **Step Computation: Approximate a solution,** s_k, **to the subproblem**

$$\min_{s \in X} \; m_k(s) \quad \textbf{subject to} \quad \|s\|_X \leq \Delta_k.$$

3. **Objective Update: Choose a new** $\widehat{J}_k(x) \approx \widehat{J}(x)$
4. **Step Acceptance: Compute**

$$\rho_k = \frac{\widehat{J}_k(x_k) - \widehat{J}_k(x_k + s_k)}{m_k(0) - m_k(s_k)}$$

if $\rho_k \geq \eta \in (0, 1)$, **then** $x_{k+1} = x_k + s_k$ **else** $x_{k+1} = x_k$.

5. **Trust Region Update: Choose a new trust region radius** Δ_{k+1}.

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Inexact Trust Region Framework

To obtain global convergence, we require

1. Inexact gradient condition

$$\|\nabla m_k(0) - \nabla \widehat{J}(x_k)\| \leq \kappa \min\{\|\nabla m_k(0)\|_X, \Delta_k\}$$

   where $\kappa > 0$ is independent of k.

2. Inexact objective condition

$$|\widehat{J}(x_k) - \widehat{J}(x_k + s_k) - (\widehat{J}_k(x_k) - \widehat{J}_k(x_k + s_k))| \leq K\theta_k$$
$$\theta_k^\omega \leq \eta \min\{m_k(0) - m_k(s_k), r_k\}$$

   for some $K > 0$, η is an algorithmic parameter and $r_k \geq 0$ with $\lim_{k\to\infty} r_k = 0$

Observation: We cannot compute $\widehat{J}(x_k)$ and $\nabla \widehat{J}(x_k)$. We use adaptive risk-informed sampling.

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Inexact TR and Adaptive RB

- From our local RB method, we obtain surrogates of $S(\xi; z)$, denoted by $S_{\text{mod}}(\xi; z)$ and $S_{\text{obj}}(\xi; z)$, to build m_k and $\widehat{J}_k$, respectively.
- Computable a posterior error indicators for the surrogate models.

$$\|S_{\text{mod}}(\xi; z) - S(\xi; z)\|_U \lesssim \epsilon^u_{\text{mod}}(\xi; z)$$
$$\|S_{\text{obj}}(\xi; z) - S(\xi; z)\|_U \lesssim \epsilon^u_{\text{obj}}(\xi; z).$$

- Local RB for the adjoint ($\Lambda_{\text{mod}}(\xi; z)$) and its computable a posterior error indicator:

$$\|\Lambda_{\text{mod}}(\xi; z) - \Lambda(\xi; z)\|_U \lesssim \epsilon^\lambda_{\text{mod}}(\xi; z).$$

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Inexact TR and Adaptive RB (2)

- **Local Reduced Basis Models:** $S_{\text{mod}}(\xi; z)$, $S_{\text{obj}}(\xi; z)$, $\Lambda_{\text{mod}}(\xi; z)$
- **Inexact Gradient Condition:**

$$\|\nabla \widehat{J}(x_k) - \nabla m_k(0)\| \lesssim \sum_{j=1}^{m} p_j \delta_{\text{mod}}(\xi_j; x_k)$$

$$\delta_{\text{mod}}(\xi_j; x_k) := \mathcal{F}(\epsilon^u_{\text{mod}}(\xi_j; x_k), \epsilon^\lambda_{\text{mod}}(\xi_j; x_k))$$

- **Inexact Objective Condition:**

$$|\widehat{J}_k(x_k) - \widehat{J}(x_k)| \lesssim \sum_{j=1}^{m} p_j \delta_{\text{obj}}(\xi_j; x_k)$$

$$\delta_{\text{obj}}(\xi_j; x_k) := \mathcal{G}(\epsilon^u_{\text{obj}}(\xi; x_k))$$

We can satisfy these conditions by controlling the Local RB errors.

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NUMERICAL EXAMPLES

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Advection-Diffusion Control

We consider the optimal control problem

$$\min_{z \in L^2(0,1)} \frac{1}{2}\mathrm{CVaR}_\beta\left(\|DS(z) - d\|_Y^2\right) + \frac{\alpha}{2}\int_0^1 z^2\,dx$$

where $u = S(z)$ solves the weak form of

$$-\epsilon\frac{\partial^2 u}{\partial x^2}(x,\omega) + b(x,\omega)\frac{\partial u}{\partial x}(x,\omega) = z(x), \quad x \in (0,1), \quad \text{a.s.}$$

$$u(0,\omega) = u(1,\omega) = 0, \qquad \text{a.s.}$$

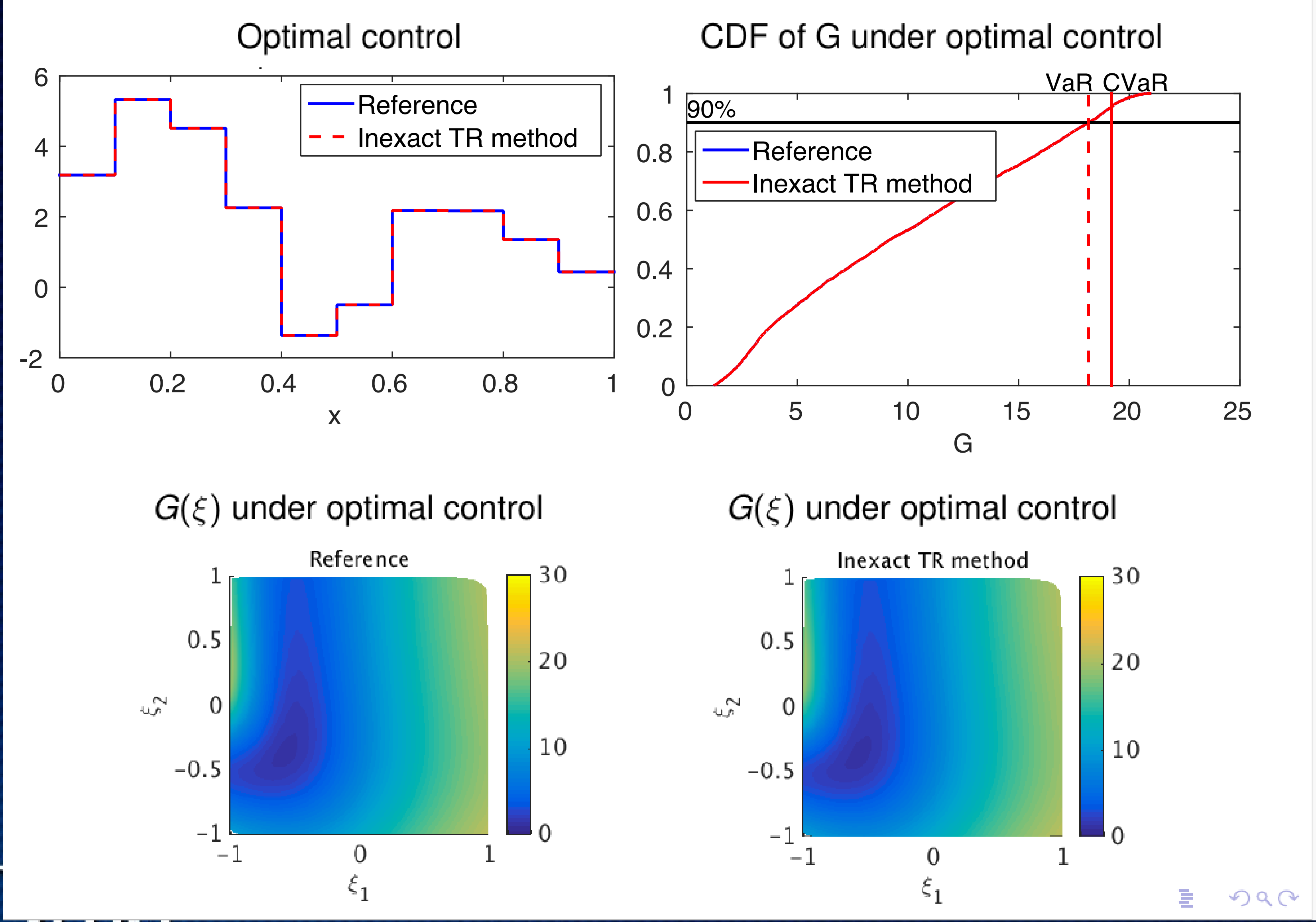
Here ϵf are deterministic and b is the random field

$$b(x,\omega) = [0.5 + \xi_1(\omega)]\,\mathbb{1}_{[0,0.5)} + [0.8 + \xi_2(\omega)]\,\mathbb{1}_{[0.5,1]}.$$

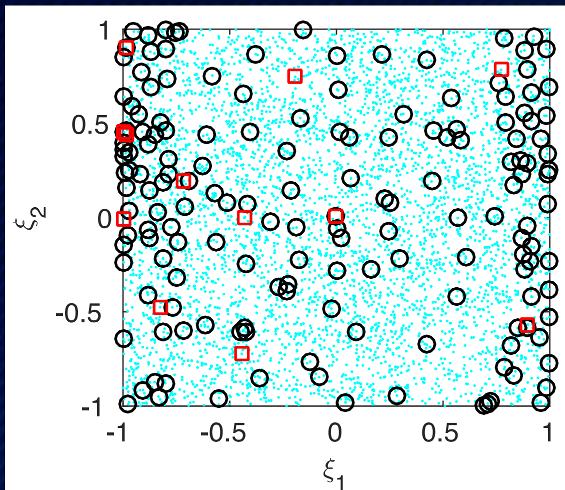
Control Ansatz:

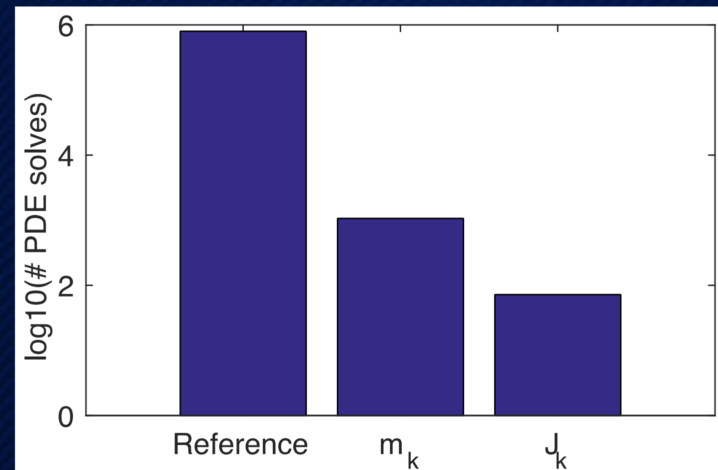
$$z = \sum_{i=1}^{10} z_i \mathbb{1}_{D_i} \quad \text{where} \quad D_i = (0.1(i-1), 0.1i)$$

Results for 1D Advection-Diffusion



Results for 1D Advection-Diffusion





Evolution of Sampling over Optimization Run

Summary

- These are the main ingredients:
  - Local basis enriched with gradient information
  - A practical and effective error indicator
  - Inexact trust region framework
- The cost of the reduced bases solutions does not grow with the number of atoms added.
- We are in the process of extending this framework to nonlinear PDEs.

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Current and Future Directions

- Treatment of physical and stochastic dimensions in a unified framework
- Model-form uncertainty
- Treatment of imperfect knowledge on underlying probability laws.

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Adaptive Algorithm 1

Algorithm 1: Adaptive algorithm to build S_{mod} and Λ_{mod} for m_k

If $k = 0$, model initialization:

- Let $\xi_0 = \mathbb{E}[\xi]$ and build the initial surrogate models S_{mod} and Λ_{mod} based on the solution $S(\xi_0; z_0)$, its gradient $\nabla_\xi S(\xi_0; z_0)$, the adjoint solution $\Lambda(\xi_0; S(\xi_0; z_0))$ and its gradient $\nabla_\xi \Lambda(\xi_0; S(\xi_0; z_0))$.

Model refinement:

Given w_k, Δ_k and S_{mod}, which is recycled from step $k-1$,

- Build m_k with S_{mod}, evaluate $\delta_{\text{mod}}(\xi_j; w_k)$ for all $j = 1, \ldots, N$, compute $E_{\text{mod}}(w_k)$ and $\|\nabla m_k(0)\|_Z$.
- **While** $E_{\text{mod}}(w_k) > \kappa \min\{\|\nabla m_k(0)\|_Z, \Delta_k\}$, **do**
  - Select $\xi_{\max} = \arg\max_{j=1,\ldots,N} p_j \delta_{\text{mod}}(\xi_j; w_k)$.
  - Compute $S(\xi_{\max}; z_k)$, $\Lambda(\xi_{\max}; S(\xi_{\max}; z_k))$, $\nabla_\xi S(\xi_{\max}; z_k)$ and $\nabla_\xi \Lambda(\xi_{\max}; S(\xi_{\max}; z_k))$.
  - Incorporate the new information at $\xi_{\max}$ into S_{mod} and Λ_{mod} using the local RB method.
  - Update m_k with S_{mod} and Λ_{mod}.
  - Update $\delta_{\text{mod}}(\xi_j; w_k)$ for $j = 1, \ldots, N$ and $E_{\text{mod}}(w_k)$, and recompute $\|\nabla m_k(0)\|_Z$.

  End

Return S_{mod} and Λ_{mod}.

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Adaptive Algorithm 2

Algorithm 2: Adaptive algorithm to build S_{obj} for $\widehat{J}_k$

If $k = 0$**, model initialization**:

- Let $\xi_0 = \mathbb{E}[\xi]$ and build the initial surrogate model S_{mod} based on the solution $S(\xi_0, z_0)$, the gradient $\nabla_\xi S(\xi_0, z_0)$.

Else, model refinement:

Given w_k, s_k, pred_k, r_k and S_{obj}, which is recycled from step $k-1$,

- Compute $\gamma_k = K\left(\eta \min\left\{\text{pred}_k, r_k\right\}\right)^{\frac{1}{\omega}}$, evaluate $\delta_{\text{obj}}(\xi_j; w_k)$, $\delta_{\text{obj}}(\xi_j; w_k + s_k)$ for $j = 1, \ldots, N$ and compute $E_{\text{obj}}(w_k)$ and $E_{\text{obj}}(w_k + s_k)$.
- **While** $E_{\text{obj}}(w_k) + E_{\text{obj}}(w_k + s_k) > \gamma_k$, **do**
  - Select $\xi_{\max} = \arg\max_{j=1,\ldots,N} p_j\left(\delta_{\text{obj}}(\xi_j; w_k) + \delta_{\text{obj}}(\xi_j; w_k + s_k)\right)$.
  - Compute $S(\xi_{\max}, z_k)$, $\nabla_\xi S(\xi_{\max}, z_k)$, $S(\xi_{\max}, z_k + \zeta_k)$, and $\nabla_\xi S(\xi_{\max}, z_k + \zeta_k)$.
  - Incorporate the new information at $\xi_{\max}$ into S_{obj} using the local RB method.
  - Update $\delta_{\text{mod}}(\xi_j; w_k)$ and $\delta_{\text{obj}}(\xi_j; w_k + s_k)$ for $j = 1, \ldots, N$, update $E_{\text{obj}}(w_k)$ and $E_{\text{obj}}(w_k + s_k)$.

  End

End

Return S_{obj}.

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Basic Definitions

Consider a function $F(X, z)$, where X is random and z is deterministic.

- The cummulative distribution of F is defined as

$$\psi(x, z) = P(F(X, z) \leq x)$$

- The β Value-at-Risk (β-VaR)

$$x^{\beta}_{VaR} := \min\{x : \psi(x, z) \geq \beta\}$$

- The β Conditional-Value-at-Risk (β-CVaR)

$$x^{\beta}_{CVaR} := E\left[F(X, z) | F(X, z) \geq x^{\beta}_{VaR}\right]$$

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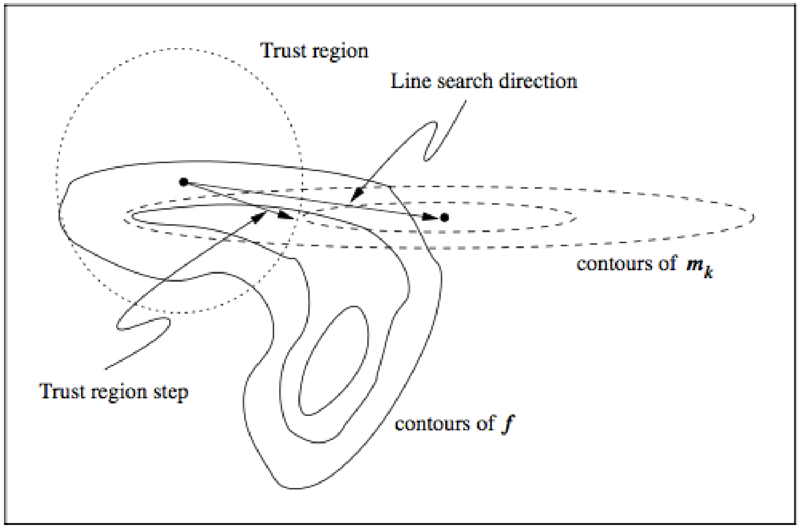
Conventional Trust Region Algorithm

Given $\bar{\Delta} > 0$, $\Delta_0 \in (0, \bar{\Delta})$, and $\eta \in \left[0, \frac{1}{4}\right)$:
for $k = 0, 1, 2, \ldots$
  Obtain p_k by (approximately) solving (4.3);
  Evaluate ρ_k from (4.4);
  if $\rho_k < \frac{1}{4}$
    $\Delta_{k+1} = \frac{1}{4}\|p_k\|$
  else
    if $\rho_k > \frac{3}{4}$ and $\|p_k\| = \Delta_k$
      $\Delta_{k+1} = \min(2\Delta_k, \bar{\Delta})$
    else
      $\Delta_{k+1} = \Delta_k$;
  if $\rho_k > \eta$
    $x_{k+1} = x_k + p_k$
  else
    $x_{k+1} = x_k$;
end (for).



From Numerical Optimization, Nocedal and Wright, Second ed.

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Numerical Example

- The regularization term is taken as

$$\wp(z) = \alpha\|z\|_Z^2, \;\; \alpha > 0$$

- The control $z(x)$ is assumed to be piecewise constants, i.e., $z(x) = \sum_{i=1}^{10} z_i \mathbf{1}_{I_i}(x)$ where $I_i = (0.1(i-1), 0.1i)$, $i = 1, \dots, 10$.
- We used the following TR parameters: $\eta_1 = 0.05$, $\eta_2 = 0.75$, $\gamma = 0.5$, $\omega = 0.75$, $\kappa = 0.1$, $K = 1$ and $\Delta_0 = 1$.
- We set the penalty parameter to $\alpha = 0.1$ and the number of Monte Carlo samples to $N = 4000$.

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