A Posteriori Error Indicator

Define the surrogate error as \( e(x, \xi) = u^h(x, \xi) - \bar{u}(x, \xi) \in U^h \), which satisfies

\[
a(e, v^h; \xi) = r(\bar{u}, v^h; \xi) \quad \forall v^h \in V^h
\]

where \( r(\bar{u}, v^h; \xi) = l(v^h) - a(\bar{u}, v^h; \xi) \) is the residual. \( r(\bar{u}, \cdot; \xi) \in V^{h'} \), by Riesz representation theorem, there exists \( \tilde{e} \in V^h \) such that

\[
r(\bar{u}, v^h; \xi) = (\tilde{e}, v^h)_V \quad \forall v^h \in V^h
\]

\[
\| r(\bar{u}, \cdot; \xi) \|_{V'} = \| \tilde{e} \|_V
\]

hence \( a(e, v^h; \xi) = (\tilde{e}, v^h)_V \quad \forall v^h \in V^h \). By the discrete inf-sup condition,

\[
\gamma^h(\xi) \| e \|_U \leq \sup_{v^h \in V^h} \frac{a(e, v^h; \xi)}{\| v^h \|_V} = \sup_{v^h \in V^h} \frac{(\tilde{e}, v^h)_V}{\| v^h \|_V} \leq \| \tilde{e} \|_V = \| r(\bar{u}, \cdot; \xi) \|_{V'}
\]

that is,

\[
\| e \|_U \leq \frac{\| r(\bar{u}, \cdot; \xi) \|_{V'}}{\gamma^h(\xi)}
\]
A Posteriori Error Indicator (2)

A candidate for the a posterior error estimator is

\[ \varepsilon_u^0(\xi) := \frac{\| r(\tilde{u}, \cdot; \xi) \|_{V'}}{\gamma^h(\xi)} \]

An upper bound on \( \| e \|_U \), could be conservative.

\[ \| e(\xi) \|_U \leq \varepsilon_u^0(\xi). \]

**Problem**: expensive to compute \( \gamma^h(\xi) \).

**Solution**: develop an efficient surrogate for \( \gamma^h(\xi) \), denoted by \( \hat{\lambda}(\xi) \).
A Posteriori Error Indicator (3)

- We used an adaptive sparse grid approach to approximate $\hat{\lambda}(\xi)$.

- Given $\hat{\lambda}(\xi)$, we have a practical error estimator as

$$
\epsilon_u(\xi) := \frac{\| r(\bar{u}, \cdot; \xi) \|_{V'}}{\theta \hat{\lambda}(\xi)}
$$

- Even with an inaccurate $\hat{\lambda}(\xi)$, by choosing $\theta \in (0, 1)$, we can still bound the error as

$$
\| e(\xi) \|_U \leq \epsilon_u(\xi)
$$
Adaptive Algorithm

**Initialization:** \[ \eta_k = p_k \mathbb{E} [\epsilon_u(\xi) \mid \xi \in \Xi_k] \]

- Specify the maximum number of atoms \( \overline{m} \) and the error tolerance \( \text{Err}_{\text{tol}} \).
- Form a background set \( \Xi_{\text{bkg}} := \{\xi_i, i = 1, 2, \ldots, N_{\text{bkg}}\} \) consisting of a sufficiently large number of Monte Carlo samples of \( \xi \).
- Form an initial training set \( \Xi_{\text{tr}} := \{\xi_i, i = 1, 2, \ldots, N_{\text{tr}}^{\text{init}}\} \) consisting of a few random samples of \( \xi \).
- Select an initial atom \( \Theta = \{\Xi[\xi]\} \) and build the initial surrogate model \( \overline{u}(x, \xi) \)

**while** \( N_{\text{atom}} < \overline{m} \) **and** \( e_{\text{max}} > e_{\text{tol}} \) **do**

Evaluate the error indicator \( \epsilon_u(\xi) \) at each \( \xi \in \Xi_{\text{tr}} \);

Compute \( \eta_k, k = 1, 2, \ldots, N_{\text{atom}} \) via implicit Voronoi tessellation of \( \Xi \);

Set \( \bar{k} = \arg \max_{k=1,2,\ldots,N_{\text{atom}}} \eta_k \), \( e_{\text{max}} = \max_{k=1,2,\ldots,N_{\text{atom}}} \eta_k \);

Set \( \xi_{\text{max}} = \arg \max_{\xi \in \Xi_{\text{tr}} \cap \Xi_{\overline{k}}} \epsilon_u(\xi) \), \( \Theta = \Theta \cup \{\xi_{\text{max}}\} \) and \( N_{\text{atom}} = N_{\text{atom}} + 1 \);

Incorporate new information at \( \{\xi_{\text{max}}\} \) into the surrogate \( \overline{u}(x, \xi) \);

Draw \( N_{\text{tr}}^{\text{add}} \) samples from \( \Xi_{N_{\text{atom}}} \) and append them into \( \Xi_{\text{tr}} \);

Draw additional training samples to ensure \( \min_{k=1,2,\ldots,N_{\text{atom}}} |\Xi_{\text{tr}} \cap \Xi_k| \geq N_{\text{tr}}^{\text{add}} \).

**end**
NUMERICAL RESULTS
1D Helmholtz Problem with Two Stochastic Dimensions

\[- \frac{d}{dx} \left( \nu(x, \omega) \frac{du}{dx} \right) - i c \tau u - \tau^2 u + f = 0, \quad (x, \omega) \in D \times \Omega \text{ a.s.} \]

\[u(0) = u(1) = 0, \quad \text{a.s.}\]

The Young’s modulus of the bar is modeled as

\[\nu(x, \omega) = [1 + 3 \xi_1(\omega)] 1(x \in [0, 0.5)) + [2.5 + 3 \xi_2(\omega)] 1(x \in [0.5, 1]).\]

where \( \xi_1 \sim \text{Beta}(1, 3) \) and \( \xi_2 \sim \text{Beta}(3, 2) \) with a correlation \( \rho(\xi_1, \xi_2) = 0.5 \). The angular frequency \( \tau = 2\pi \).
1D Helmholtz Results
Norm of the Solution in Parameter Space

c = 0.25
c = 0.01
1D Helmholtz Results: Partitions generated

Partitions generated with 5, 20, and 40 atoms and $c=0.25$
1D Helmholtz Problem
Relative error statistics using 1000 MC samples

\[ e_u(\xi) = \frac{\| \bar{u}(\xi) - u(\xi) \|_{L_2(\mathcal{D})}}{\| u(\xi) \|_{L_2(\mathcal{D})}} \]
2D Advection-Diffusion Example
31 Stochastic Dimensions

\[ -\nabla \cdot \kappa(x, \omega) \nabla u(x, \omega) + \mathbf{v}(x, \omega) \cdot \nabla u(x, \omega) + f(x, \omega) = 0 \]
\[ u(x, \omega) = 0 \text{ on } \Gamma_D \]
\[ \kappa(x, \omega) \nabla u(x, \omega) \cdot \mathbf{n} = 0 \text{ on } \Gamma_N \]

\[ u(x, 0) = 0 \]

\[ \kappa = \sum_{i}^{5} \sum_{j}^{5} \phi_i(x_1)\phi_j(x_2)\xi_{5(i-1)+j} \]

where \( \phi_i(x) = B_{i,5}(x) \) is a Bernstein’s polynomial of order 5.
\( \xi_k \) are beta-distributed, independent random variables

\[ \mathbf{v}(x, \omega) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_{26} + \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} \xi_{27} \]

\( \xi_{26} \sim U[10, 20] \quad \xi_{27} \sim U[0, 10] \)
2D Advection-Diffusion Results
Comparison with Sparse Grid Methods

For 5580 FE solves
1D Advection-Diffusion Example

2 Stochastic Dimensions

\[ u(0) = 0 \quad f(x) = 1, \quad a = 0.1 \]
\[ b_1^0 + b_1(\omega) \quad b_2^0 + b_2(\omega) \]
\[ u(1) = 0 \]

\[-a \frac{d^2 u(x, \omega)}{d x^2} + b(x, \omega) \frac{d u(x, \omega)}{d x} + f(x) = 0 \quad (x, \omega) \in [0, 1] \times \Omega \]
\[ u(0) = 0 \]
\[ u(1) = 0 \]

\[ b(x, \omega) = \left( b_1^0 + b_1(\omega) \right) 1(x \in [0, 0.5)) + \left( b_2^0 + b_2(\omega) \right) 1(x \in [0.5, 1]) \]

where \( b_i(\omega) \sim U([-1, 1]) \) and independent

In this example, \( b_1^0 = 0.5, b_2^0 = 0.8 \). The variable \( b_1(\omega) \) is more influential than \( b_2(\omega) \) on the solution. The latter is true especially when the value of \( b_1(\omega) \) is small.
1D Advection-Diffusion Results
Partitions Generated

Partition generated using input (i.e. no adaptivity)

Partition generated using adaptivity on the output
1D Advection-Diffusion Results
Moments using two approaches

5 Atoms

20 Atoms
1D Advection-Diffusion: Joint densities at $x=0.9$ and $x=0.1$

5 Atoms

20 Atoms
1D Advection-Diffusion
Relative error statistics using 1000 MC samples

\[ e_u(\xi) = \frac{\| \bar{u}(\xi) - u(\xi) \|_{L_2(D)}}{\| u(\xi) \|_{L_2(D)}} \]