A Posteriori Error Indicator

Define the surrogate error as $e(x,\boldsymbol{\xi}) = u^h(x,\boldsymbol{\xi}) - \bar{u}(x,\boldsymbol{\xi}) \in U^h$, which satisfies

$$a(e, v^h; \boldsymbol{\xi}) = r(\bar{u}, v^h; \boldsymbol{\xi}) \quad \forall v^h \in V^h$$

where $r(\bar{u}, v^h; \boldsymbol{\xi}) = l(v^h) - a(\bar{u}, v^h; \boldsymbol{\xi})$ is the residual. $r(\bar{u}, \cdot; \boldsymbol{\xi}) \in V^{h'}$, by Riesz representation theorem, there exists $\tilde{e} \in V^h$ such that

$$r(\bar{u}, v^h; \boldsymbol{\xi}) = (\tilde{e}, v^h)_V \quad \forall v^h \in V^h$$
$$\|r(\bar{u}, \cdot; \boldsymbol{\xi})\|_{V'} = \|\tilde{e}\|_V$$

hence $a(e, v^h; \boldsymbol{\xi}) = (\tilde{e}, v^h)_V \quad \forall v^h \in V^h$. By the discrete inf-sup condition,

$$\gamma^h(\boldsymbol{\xi})\|e\|_U \leq \sup_{v^h \in V^h} \frac{a(e, v^h; \boldsymbol{\xi})}{\|v^h\|_V} = \sup_{v^h \in V^h} \frac{(\tilde{e}, v^h)_V}{\|v^h\|_V} \leq \|\tilde{e}\|_V = \|r(\bar{u}, \cdot; \boldsymbol{\xi})\|_{V'}$$

that is,

$$\|e\|_U \leq \frac{\|r(\bar{u}, \cdot; \boldsymbol{\xi})\|_{V'}}{\gamma^h(\boldsymbol{\xi})}$$

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A Posteriori Error Indicator (2)

A candidate for the a posterior error estimator is

$$\epsilon_u^0(\boldsymbol{\xi}) := \frac{\|r(\bar{u}, \cdot; \boldsymbol{\xi})\|_{V'}}{\gamma^h(\boldsymbol{\xi})}$$

An upper bound on $\|e\|_U$, could be conservative.

$$\|e(\boldsymbol{\xi})\|_U \leq \epsilon_u^0(\boldsymbol{\xi}).$$

Problem: expensive to compute $\gamma^h(\boldsymbol{\xi})$.

Solution: develop an efficient surrogate for $\gamma^h(\boldsymbol{\xi})$, denoted by $\hat{\lambda}(\boldsymbol{\xi})$.

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A Posteriori Error Indicator (3)

- We used an adaptive sparse grid approach to approximate $\hat{\lambda}(\boldsymbol{\xi})$.
- Given $\hat{\lambda}(\boldsymbol{\xi})$, we have a practical error estimator as

$$\epsilon_u(\boldsymbol{\xi}) := \frac{\|r(\bar{u}, \cdot; \boldsymbol{\xi})\|_{V'}}{\theta \hat{\lambda}(\boldsymbol{\xi})}$$

- Even with an inaccurate $\hat{\lambda}(\boldsymbol{\xi})$, by choosing $\theta \in (0, 1)$, we can still bound the error as

$$\|e(\boldsymbol{\xi})\|_U \leq \epsilon_u(\boldsymbol{\xi})$$

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Adaptive Algorithm

Initialization: $\eta_k = p_k \mathbb{E}\left[\epsilon_u(\xi) \mid \xi \in \overline{\Xi}_k\right]$

- Specify the maximum number of atoms $\overline{m}$ and the error tolerance $\mathrm{Err}_{\mathrm{tol}}$.
- Form a background set $\Xi_{\mathrm{bkg}} := \{\xi_i, i = 1, 2, \ldots, N_{\mathrm{bkg}}\}$ consisting of a sufficiently large number of Monte Carlo samples of ξ.
- Form an initial training set $\Xi_{\mathrm{tr}} := \{\xi_i, i = 1, 2, \ldots, N_{\mathrm{tr}}^{\mathrm{init}}\}$ consisting of a few random samples of ξ.
- Select an initial atom $\Theta = \{\mathbb{E}[\xi]\}$ and build the initial surrogate model $\overline{u}(x, \xi)$

while $N_{\mathrm{atom}} < \overline{m}$ *and* $e_{\max} > e_{\mathrm{tol}}$ **do**

Evaluate the error indicator $\epsilon_u(\xi)$ at each $\xi \in \Xi_{\mathrm{tr}}$;

Compute $\eta_k, k = 1, 2, \ldots, N_{\mathrm{atom}}$ via implicit Voronoi tessellation of Ξ;

Set $\overline{k} = \underset{k=1,2,\ldots,N_{\mathrm{atom}}}{\arg\max}\ \eta_k$, $e_{\max} = \max_{k=1,2,\ldots,N_{\mathrm{atom}}} \eta_k$;

Set $\xi_{\max} = \underset{\xi \in \Xi_{\mathrm{tr}} \cap \overline{\Xi}_{\overline{k}}}{\arg\max}\ \epsilon_u(\xi)$, $\Theta = \Theta \cup \{\xi_{\max}\}$ and $N_{\mathrm{atom}} = N_{\mathrm{atom}} + 1$;

Incorporate new information at $\{\xi_{\max}\}$ into the surrogate $\overline{u}(x, \xi)$;

Draw $N_{\mathrm{tr}}^{\mathrm{add}}$ samples from $\overline{\Xi}_{N_{\mathrm{atom}}}$ and append them into Ξ_{tr};

Draw additional training samples to ensure $\min_{k=1,2\ldots,N_{\mathrm{atom}}} \mid \Xi_{\mathrm{tr}} \cap \overline{\Xi}_k \mid \geq N_{\mathrm{tr}}^{\mathrm{add}}$.

end

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NUMERICAL RESULTS

1D Helmholtz Problem with Two Stochastic Dimensions

$$-\frac{d}{dx}\left(\nu(x,\omega)\frac{du}{dx}\right) - ic\tau u - \tau^2 u + f = 0, \quad (x,\omega) \in D \times \Omega \text{ a.s.}$$

$$u(0) = u(1) = 0, \quad \text{a.s.}$$

The Young's modulus of the bar is modeled as

$$\nu(x,\omega) = [1 + 3\xi_1(\omega)]\,\mathbf{1}(x \in [0, 0.5)) + [2.5 + 3\xi_2(\omega)]\,\mathbf{1}(x \in [0.5, 1]).$$

where $\xi_1 \sim \text{Beta}(1,3)$ and $\xi_2 \sim \text{Beta}(3,2)$ with a correlation $\rho(\xi_1, \xi_2) = 0.5$. The angular frequency $\tau = 2\pi$.

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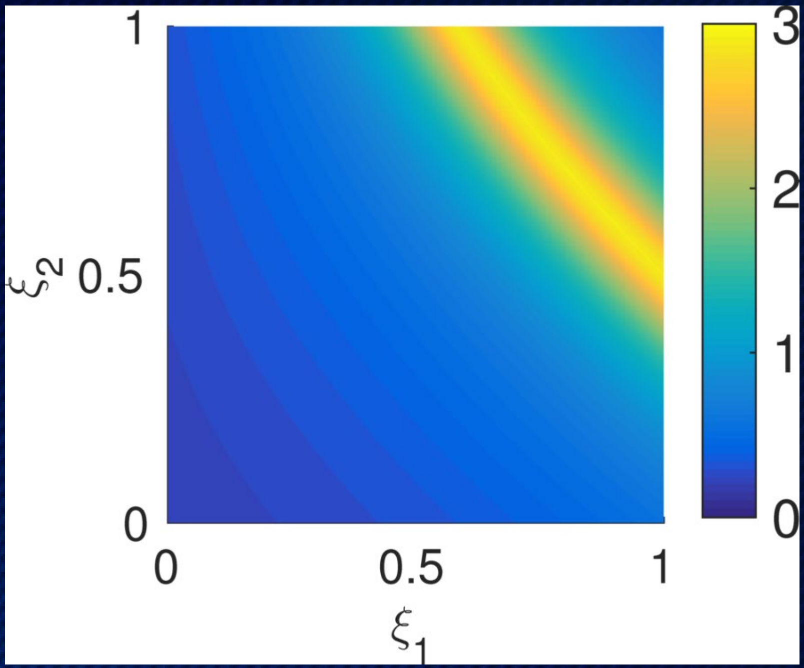
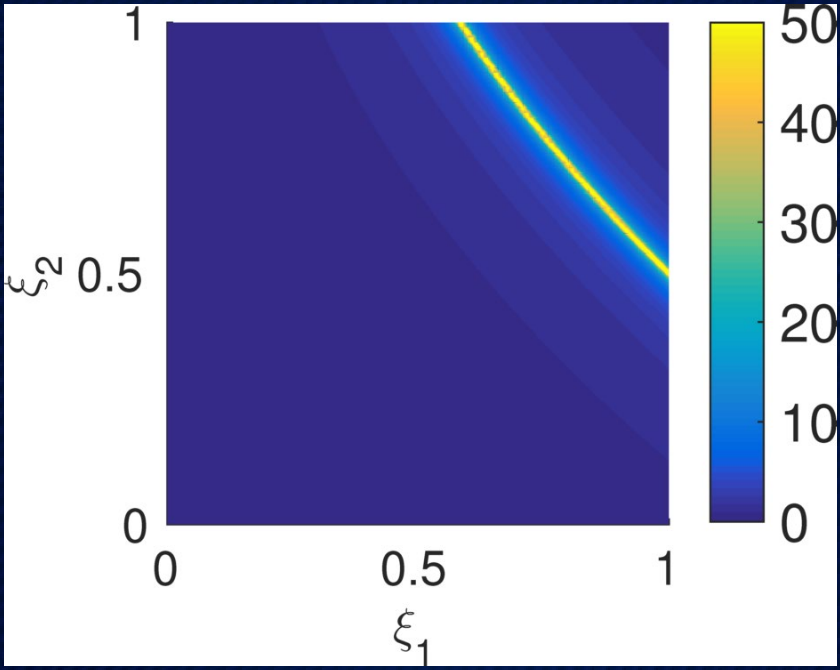
1D Helmholtz Results
Norm of the Solution in Parameter Space



c= 0.25



c= 0.01

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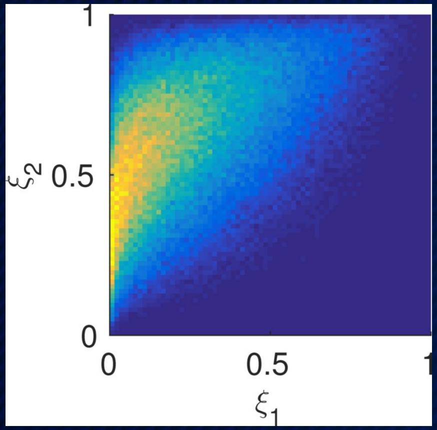
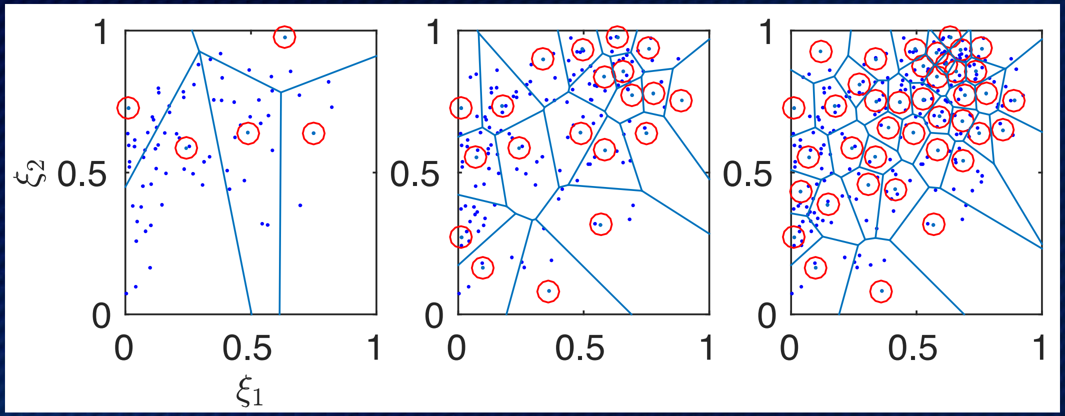
1D Helmholtz Results: Partitions generated

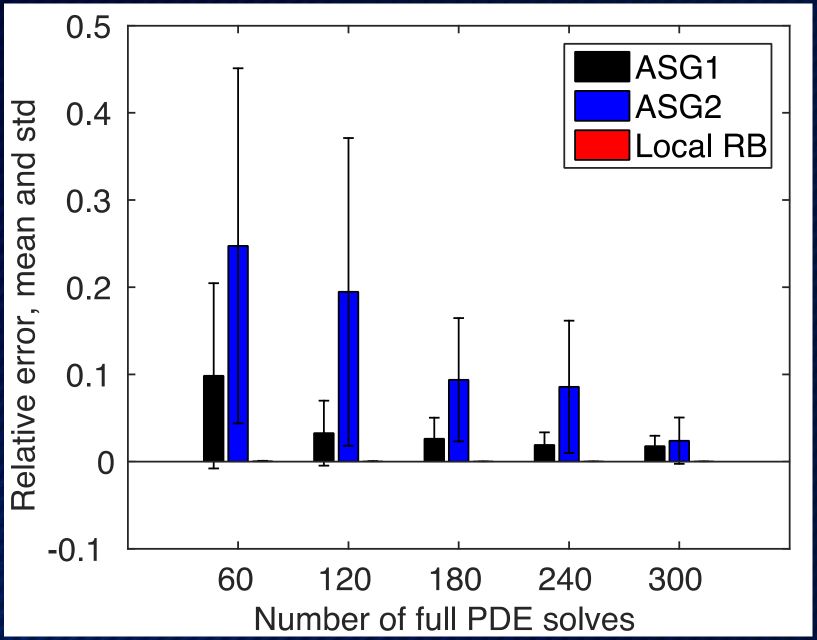


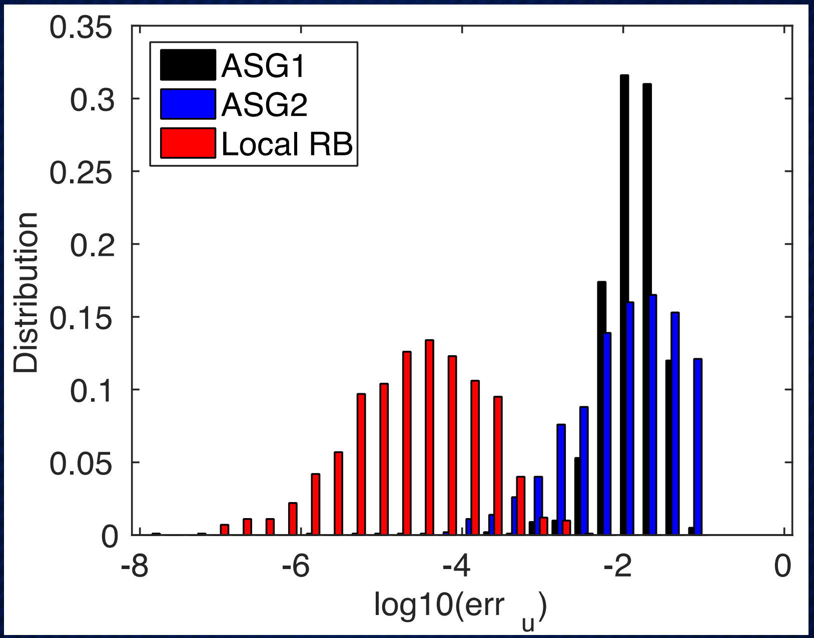
Joint parameter distribution



Partitions generated with 5, 20, and 40 atoms and c=0.25

1D Helmholtz Problem
Relative error statistics using 1000 MC samples





$$e_u(\xi) = \frac{\|\bar{u}(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\|_{L_2(\mathcal{D})}}{\|u(\boldsymbol{\xi})\|_{L_2(D)}}$$

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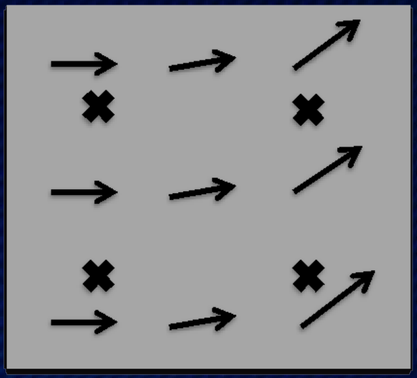
2D Advection-Diffusion Example
31 Stochastic Dimensions



$u(x,0)=0$

$$-\nabla\cdot\kappa(\boldsymbol{x},\omega)\nabla u(\boldsymbol{x},\omega)+\boldsymbol{v}(\boldsymbol{x},\omega)\cdot\nabla u(\boldsymbol{x},\omega)+f(\boldsymbol{x},\omega)=0$$
$$u(\boldsymbol{x},\omega)=0 \text{ on } \Gamma_D$$
$$\kappa(\boldsymbol{x},\omega)\nabla u(\boldsymbol{x},\omega)\cdot\boldsymbol{n}=0 \text{ on } \Gamma_N$$

$$\kappa=\sum_{i}^{5}\sum_{j}^{5}\phi_i(x_1)\phi_j(x_2)\xi_{5(i-1)+j}$$

where $\phi_i(x)=B_{i,5}(x)$ is a Bernstein's polynomial of order 5.

ξ_k are beta-distributed, independent random variables

$$\boldsymbol{v}(\boldsymbol{x},\omega)=\begin{pmatrix}1\\0\end{pmatrix}\xi_{26}+\begin{pmatrix}-x_1\\x_2\end{pmatrix}\xi_{27}$$

$$\xi_{26}\sim U[10,20]\qquad \xi_{27}\sim U[0,10]$$

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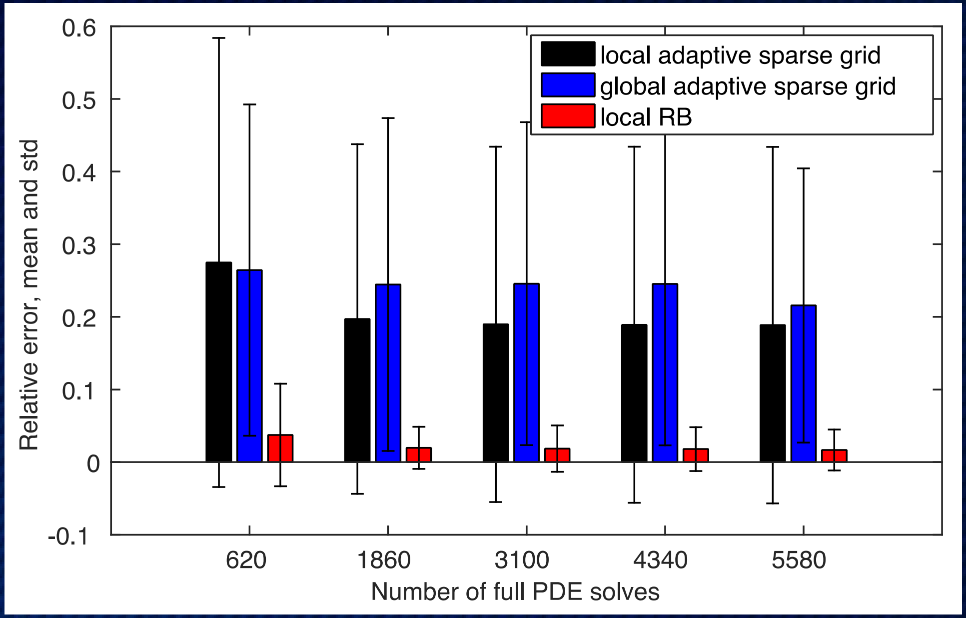
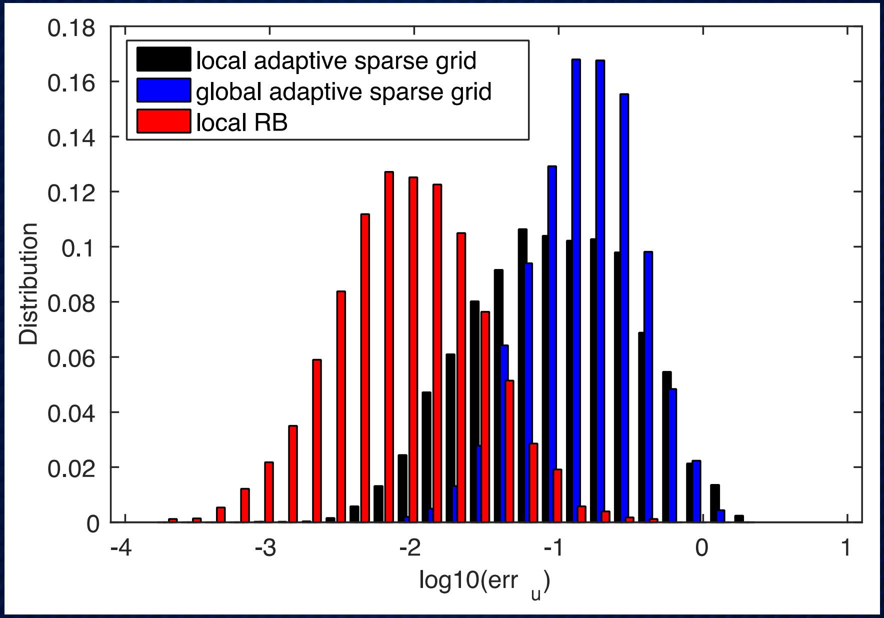
2D Advection-Diffusion Results
Comparison with Sparse Grid Methods





For 5580 FE solves

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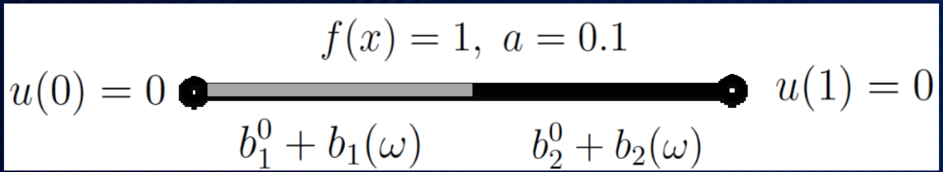
1D Advection-Diffusion Example
2 Stochastic Dimensions



$$
\begin{aligned}
-a\frac{d^2u(x,\omega)}{dx^2} + b(x,\omega)\frac{du(x,\omega)}{dx} + f(x) &= 0 \quad (x,\omega) \in [0,1]\times\Omega \\
u(0) &= 0 \\
u(1) &= 0
\end{aligned}
$$

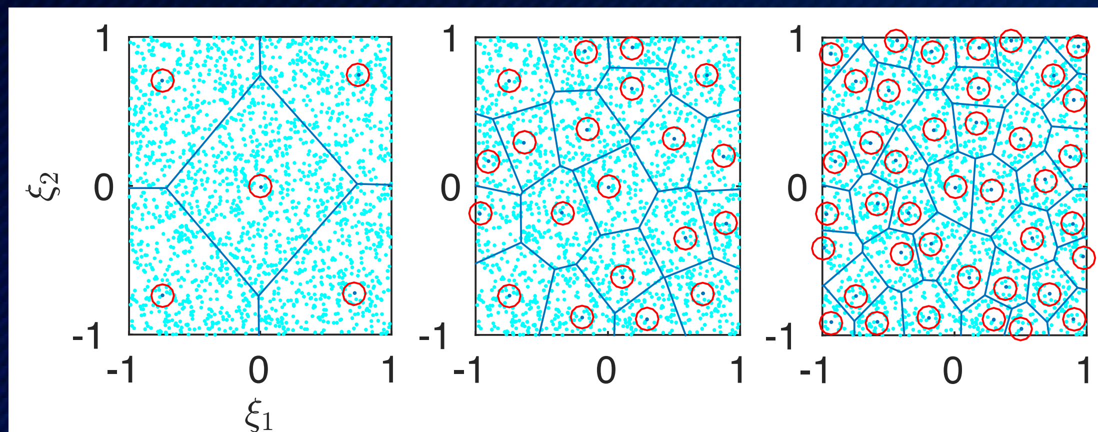
$$b(x,\omega) = \Big(b_1^0 + b_1(\omega)\Big)\mathbf{1}(x \in [0,0.5)) + \Big(b_2^0 + b_2(\omega)\Big)\mathbf{1}(x \in [0.5,1])$$

where $b_i(\omega) \sim U([-1,1])$ and independent

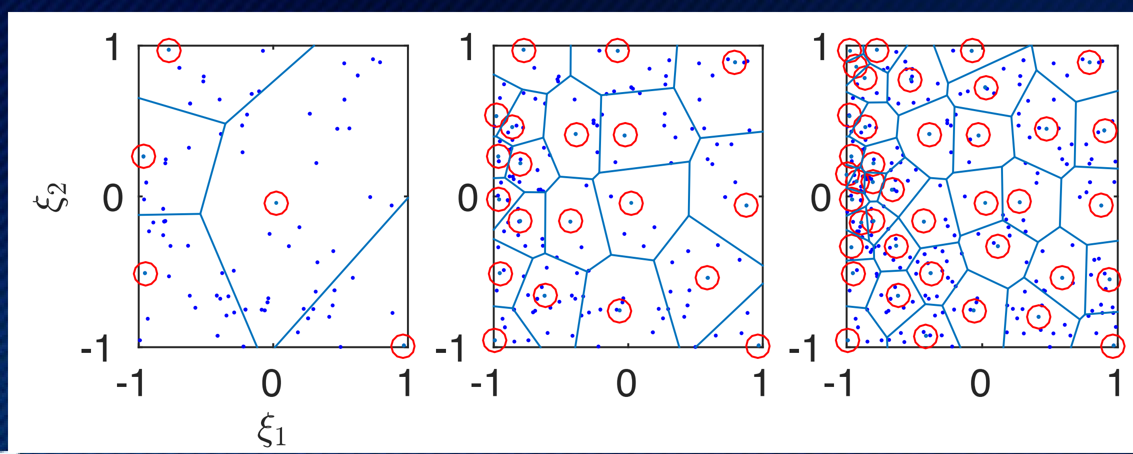
In this example, $b_1^0 = 0.5, b_2^0 = 0.8$. The variable $b_1(\omega)$ is more influential than $b_2(\omega)$ on the solution. The latter is true especially when the value of $b_1(\omega)$ is small.

1D Advection-Diffusion Results
Partitions Generated

Partition generated using input (i.e. no adaptivity)



Partition generated using adaptivity on the output



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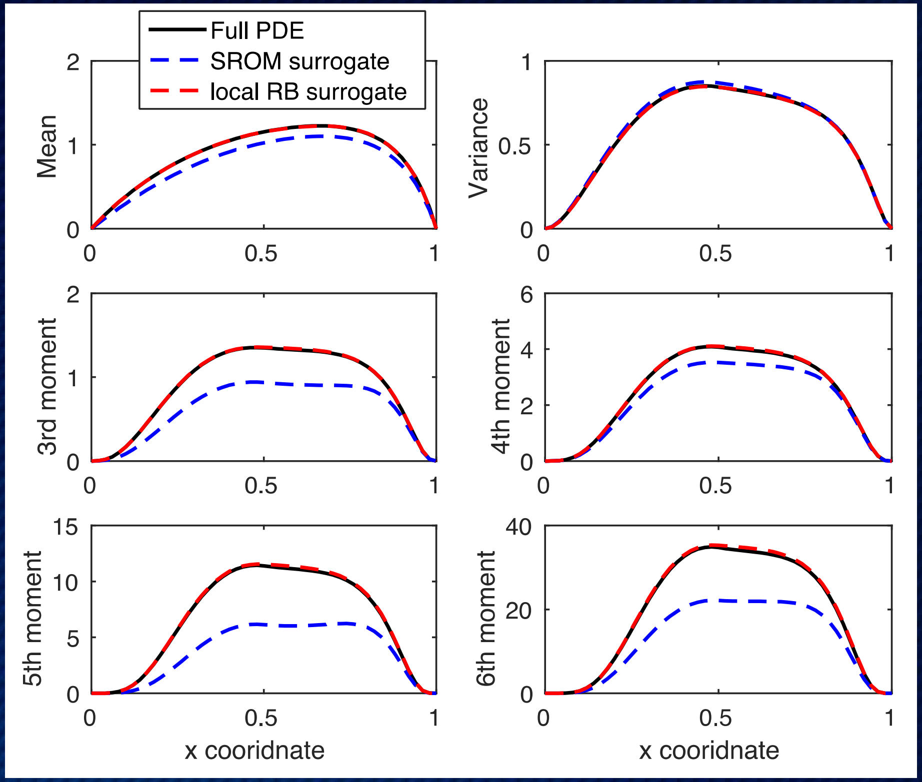
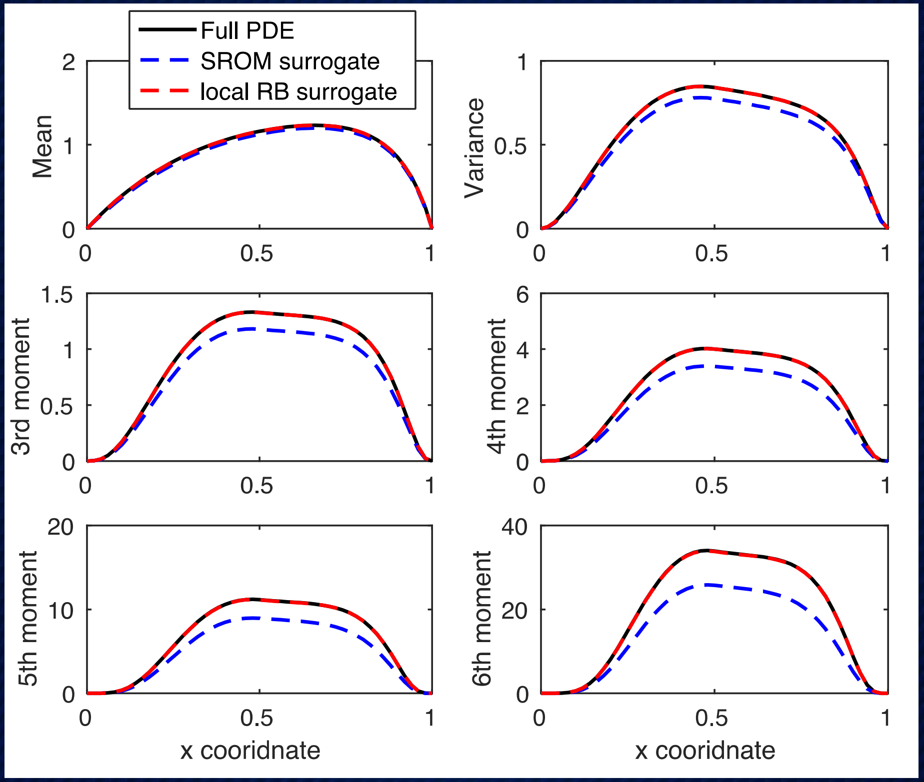
1D Advection-Diffusion Results
Moments using two approaches

5 Atoms



20 Atoms



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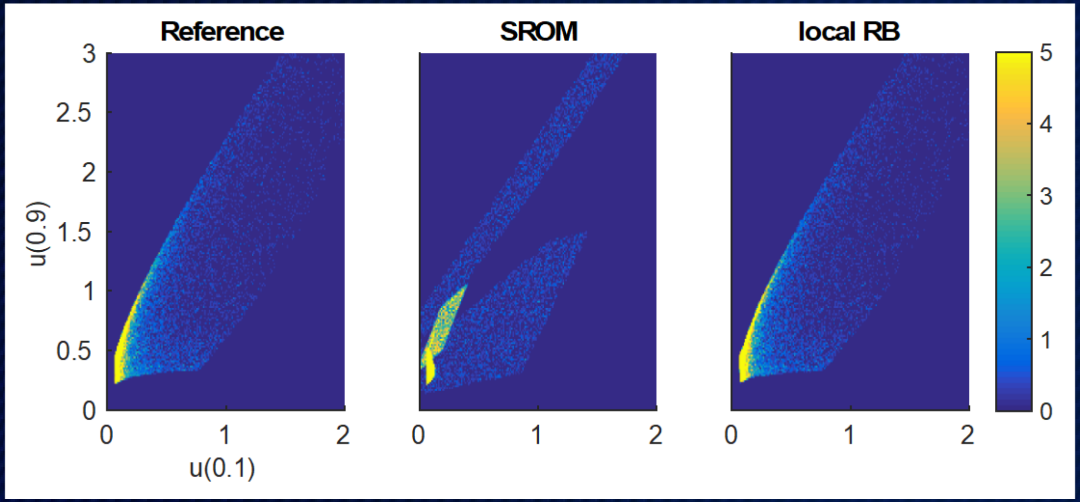
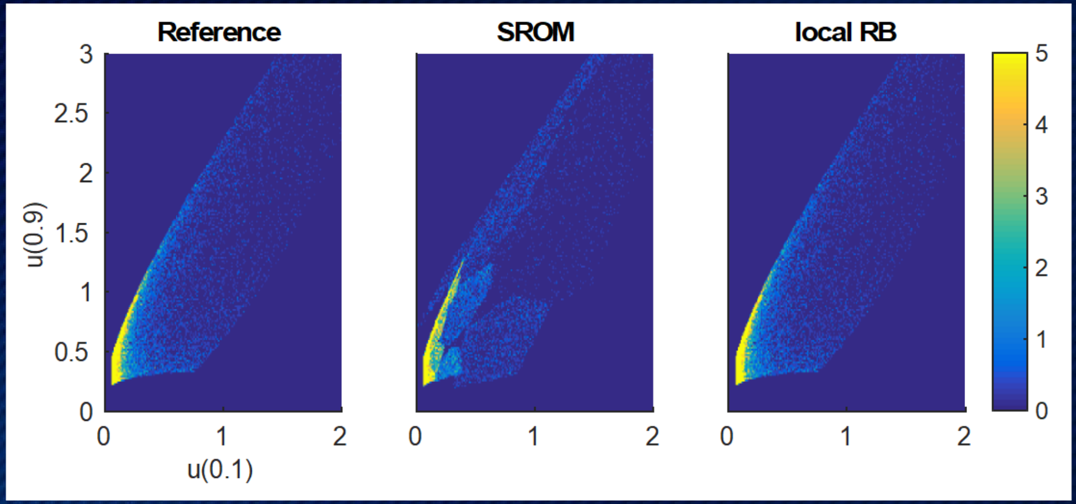
1D Advection-Diffusion: Joint densities at x=0.9 and x=0.1

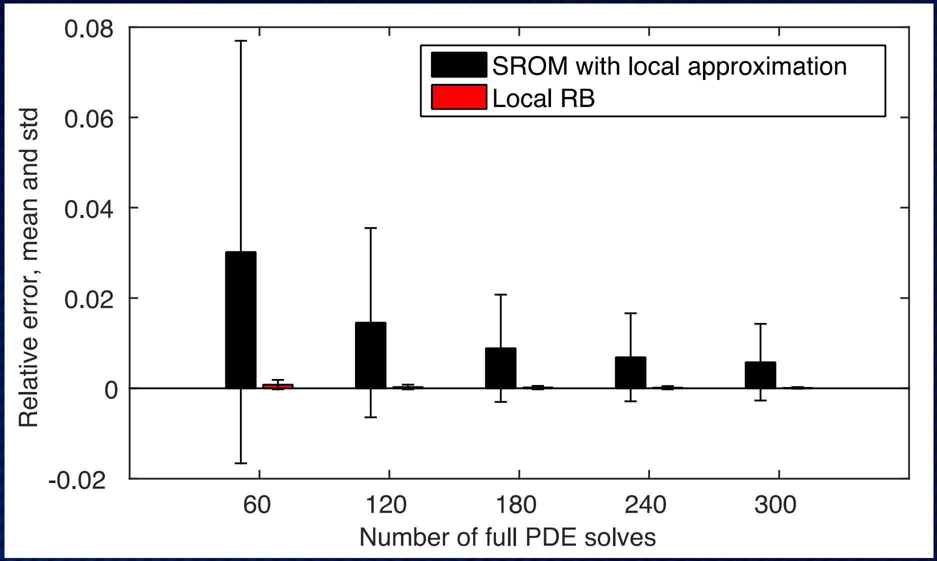
5 Atoms

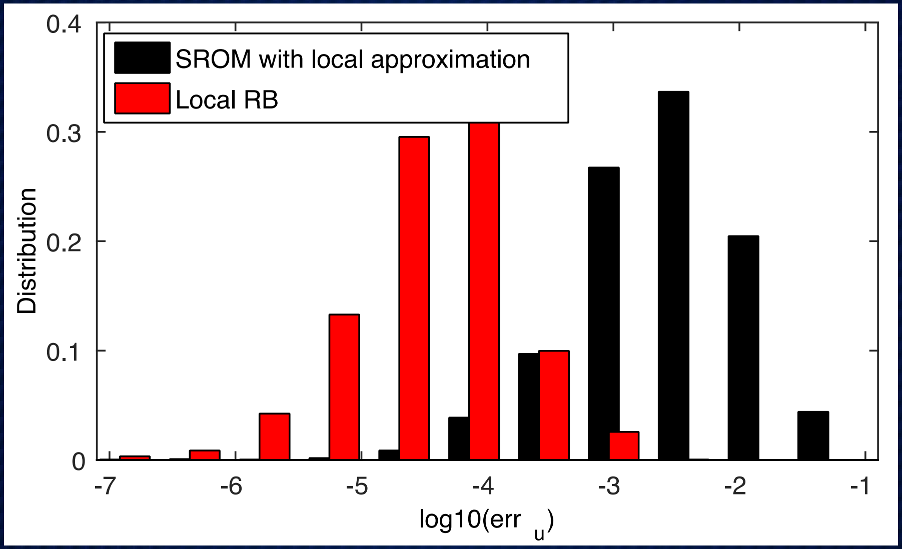


20 Atoms



1D Advection-Diffusion
Relative error statistics using 1000 MC samples





$$e_u(\xi) = \frac{\|\bar{u}(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\|_{L_2(\mathcal{D})}}{\|u(\boldsymbol{\xi})\|_{L_2(D)}}$$

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