Homogenization estimates for the macroscopic response and field statistics in viscoplastic polycrystals

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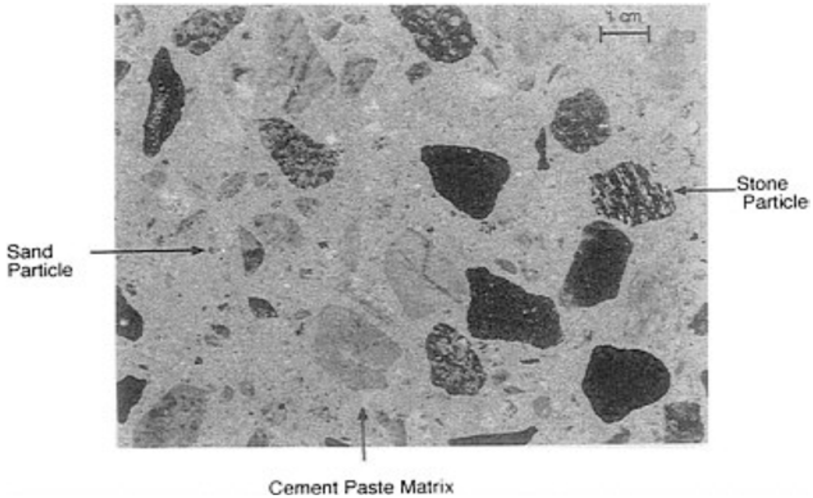
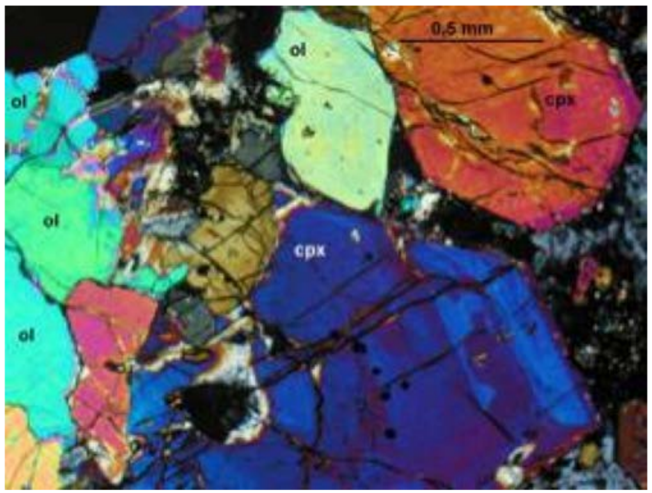
MOTIVATION

Examples of Microstructures

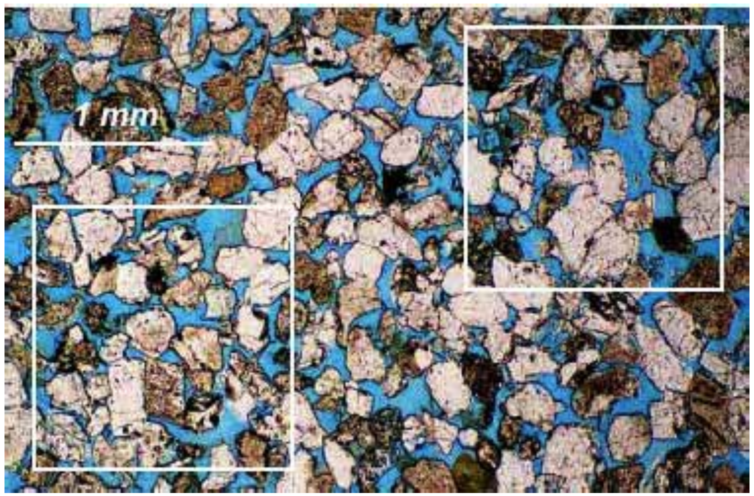


concrete

Polycrystalline ice

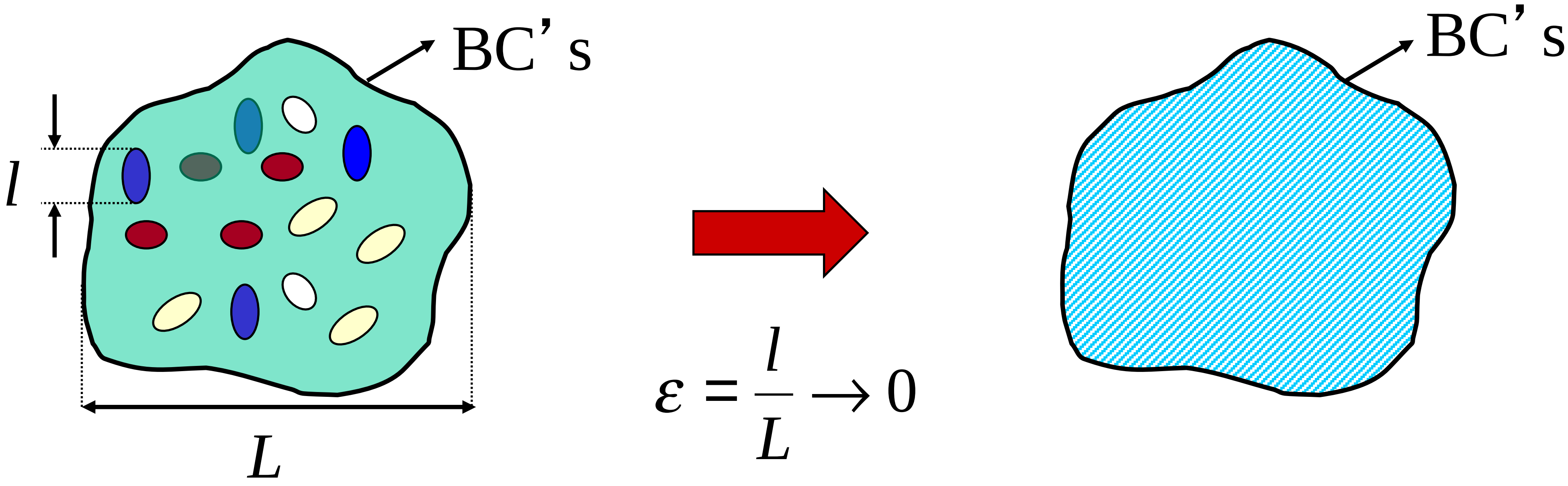


Polycrystalline olivine



Fluid-impregnated sandstone

Homogenization



$$\varepsilon = \frac{l}{L} \to 0$$

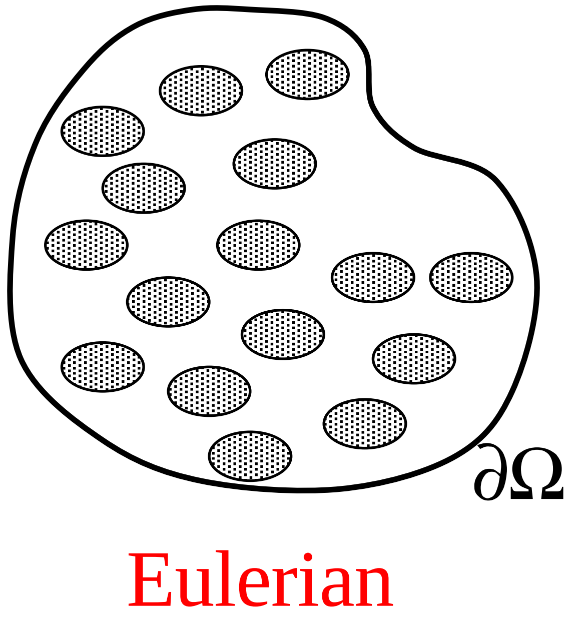
- If there is a wide separation of length scales and the boundary conditions vary "slowly" relative to the micro-scale, then an RVE of the composite behaves like a homogeneous material with effective properties depending on the properties and distribution of the constituent phases (i.e., microstructure), but not on the specific boundary conditions.

BACKGROUND

Nonlinear Elasticity/Viscoplasticity

- Strain-displacement (strain-rate/velocity) relation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$



- Equilibrium

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \quad \text{or} \quad \sigma_{ij,j} + b_i = 0$$

- Constitutive relation

$$\boldsymbol{\epsilon} = \frac{\partial u^{(r)}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$$

where $u^{(r)}$ is a convex stress potential, which may be isotropic or anisotropic

$$u^{(r)}(\boldsymbol{\sigma}) = \phi^{(r)}(\sigma_e) \qquad u^{(r)}(\boldsymbol{\sigma}) = \sum_{k=1}^{K^{(r)}} \phi_{(k)}^{(r)}(\tau_{(k)}^{(r)})$$

Homogenization: Effective Properties

- *Local* behavior:

$$\boldsymbol{\epsilon} = \frac{\partial u}{\partial \boldsymbol{\sigma}}, \qquad u(\mathbf{x}, \boldsymbol{\sigma}) = \sum_{r=1}^{N} \chi^{(r)}(\mathbf{x})\, u^{(r)}(\boldsymbol{\sigma})$$

where the $\chi^{(r)}$ characterize the microstructure.

- *Effective or macroscopic* behavior:

$$\overline{\boldsymbol{\epsilon}} = \frac{\partial \widetilde{U}}{\partial \overline{\boldsymbol{\sigma}}} \qquad \widetilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in \mathcal{S}} \langle u(\mathbf{x}, \boldsymbol{\sigma}) \rangle$$

where $\mathcal{S} = \{\boldsymbol{\sigma},\ \mathrm{div}\, \boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega,\ \boldsymbol{\sigma}\mathbf{n} = \overline{\boldsymbol{\sigma}}\mathbf{n} \text{ on } \partial\Omega\}$ and $\langle \boldsymbol{\sigma} \rangle = \overline{\boldsymbol{\sigma}}$ and $\overline{\boldsymbol{\epsilon}} = \langle \boldsymbol{\epsilon} \rangle$ are average stress and strain (rate).

Hashin (1964); Hill (1965); Hutchinson (1976); Willis (1983)

Homogenization: Field Statistics

- Phase averages:

$$\overline{\boldsymbol{\epsilon}}^{(r)} = \langle \boldsymbol{\epsilon} \rangle^{(r)} \qquad \overline{\boldsymbol{\sigma}}^{(r)} = \langle \boldsymbol{\sigma} \rangle^{(r)}$$

- Second moments/covariance tensors:

$$\langle \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} \rangle^{(r)} \quad \langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle^{(r)} \qquad \mathbb{C}_{\boldsymbol{\sigma}}^{(r)} = \langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle^{(r)} - \overline{\boldsymbol{\sigma}}^{(r)} \otimes \overline{\boldsymbol{\sigma}}^{(r)}$$

- Results can be obtained via identities of the type

$$\overline{\boldsymbol{\sigma}}^{(r)} = \frac{1}{c^{(r)}} \, \partial_{\boldsymbol{\eta}^{(r)}} \widetilde{U}_{\boldsymbol{\eta}} \Big|_{\boldsymbol{\eta}^{(r)}=\mathbf{0}}$$

$$u_{\boldsymbol{\eta}}(\mathbf{x}, \boldsymbol{\sigma}) = \sum_{s=1}^{N} \chi^{(s)}(\mathbf{x}) \, u^{(s)}(\boldsymbol{\sigma}) + \chi^{(r)}(\mathbf{x}) \, \boldsymbol{\eta}^{(r)} \cdot \boldsymbol{\sigma}$$

Bobeth & Diener (1987) JMPS

Idiart & PC (2007) PRSL A

EFFECTIVE BEHAVIOR & FIELD FLUCTUATIONS FOR 3-D, POWER-LAW POLYCRYSTALS

Liu & PC (2004) JMPS

Song & PC (2018) IJP