Homogenization estimates for the macroscopic response and field statistics in viscoplastic polycrystals

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MOTIVATION
Examples of Microstructures

- Polycrystalline ice
- Fluid-impregnated sandstone
- Polycrystalline olivine
- Concrete
If there is a wide separation of length scales and the boundary conditions vary “slowly” relative to the micro-scale, then an RVE of the composite behaves like a **homogeneous** material with **effective properties** depending on the properties and distribution of the constituent phases (i.e., **microstructure**), but **not** on the specific boundary conditions.
BACKGROUND
Nonlinear Elasticity/Viscoplasticity

- Strain-displacement (strain-rate/velocity) relation
  \[ \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]

- Equilibrium
  \[ \nabla \cdot \mathbf{\sigma} + \mathbf{b} = 0, \quad \text{or} \quad \sigma_{i,j,j} + b_i = 0 \]

- Constitutive relation
  \[ \varepsilon = \frac{\partial u^{(r)}(\mathbf{\sigma})}{\partial \mathbf{\sigma}} \]

where \( u^{(r)} \) is a convex stress potential, which may be isotropic or anisotropic

\[ u^{(r)}(\mathbf{\sigma}) = \phi^{(r)}(\mathbf{\sigma}_e) \quad u^{(r)}(\mathbf{\sigma}) = \sum_{k=1}^{K^{(r)}} \phi^{(r)}(\mathbf{\tau}^{(r)}_k) \]
Homogenization: Effective Properties

• **Local behavior:**

\[ \epsilon = \frac{\partial u}{\partial \sigma}, \quad u(x, \sigma) = \sum_{r=1}^{N} \chi^{(r)}(x) u^{(r)}(\sigma) \]

where the \( \chi^{(r)} \) characterize the microstructure.

• **Effective or macroscopic behavior:**

\[ \bar{\epsilon} = \frac{\partial \bar{U}}{\partial \bar{\sigma}}, \quad \bar{U}(\bar{\sigma}) = \min_{\sigma \in \mathcal{S}} \langle u(x, \sigma) \rangle \]

where \( \mathcal{S} = \{ \sigma, \text{ div } \sigma = 0 \text{ in } \Omega, \sigma n = \bar{\sigma} n \text{ on } \partial \Omega \} \) and \( \langle \sigma \rangle = \bar{\sigma} \) and \( \bar{\epsilon} = \langle \epsilon \rangle \) are average stress and strain (rate).

Hashin (1964); Hill (1965); Hutchinson (1976); Willis (1983)
Homogenization: Field Statistics

• Phase averages:
\[ \overline{\epsilon}(r) = \langle \epsilon \rangle(r), \quad \overline{\sigma}(r) = \langle \sigma \rangle(r) \]

• Second moments/covariance tensors:
\[ \langle \epsilon \otimes \epsilon \rangle(r) \quad \langle \sigma \otimes \sigma \rangle(r) \quad C_\sigma^{(r)} = \langle \sigma \otimes \sigma \rangle^{(r)} - \overline{\sigma}(r) \otimes \overline{\sigma}(r) \]

• Results can be obtained via identities of the type
\[ \overline{\sigma}(r) = \frac{1}{c(r)} \partial_{\eta^{(r)}} \tilde{U}_{\eta} \bigg|_{\eta^{(r)}=0} \]
\[ u_{\eta}(x, \sigma) = \sum_{s=1}^{N} \chi^{(s)}(x) u^{(s)}(\sigma) + \chi^{(r)}(x) \eta^{(r)} \cdot \sigma \]

Bobeth & Diener (1987) JMPS
Idiart & PC (2007) PRSL A
EFFECTIVE BEHAVIOR & FIELD FLUCTUATIONS FOR 3-D, POWER-LAW POLYCRYSTALS

Liu & PC (2004) JMPS

Song & PC (2018) IJP