Tensor Random Fields for Continuum Mechanics

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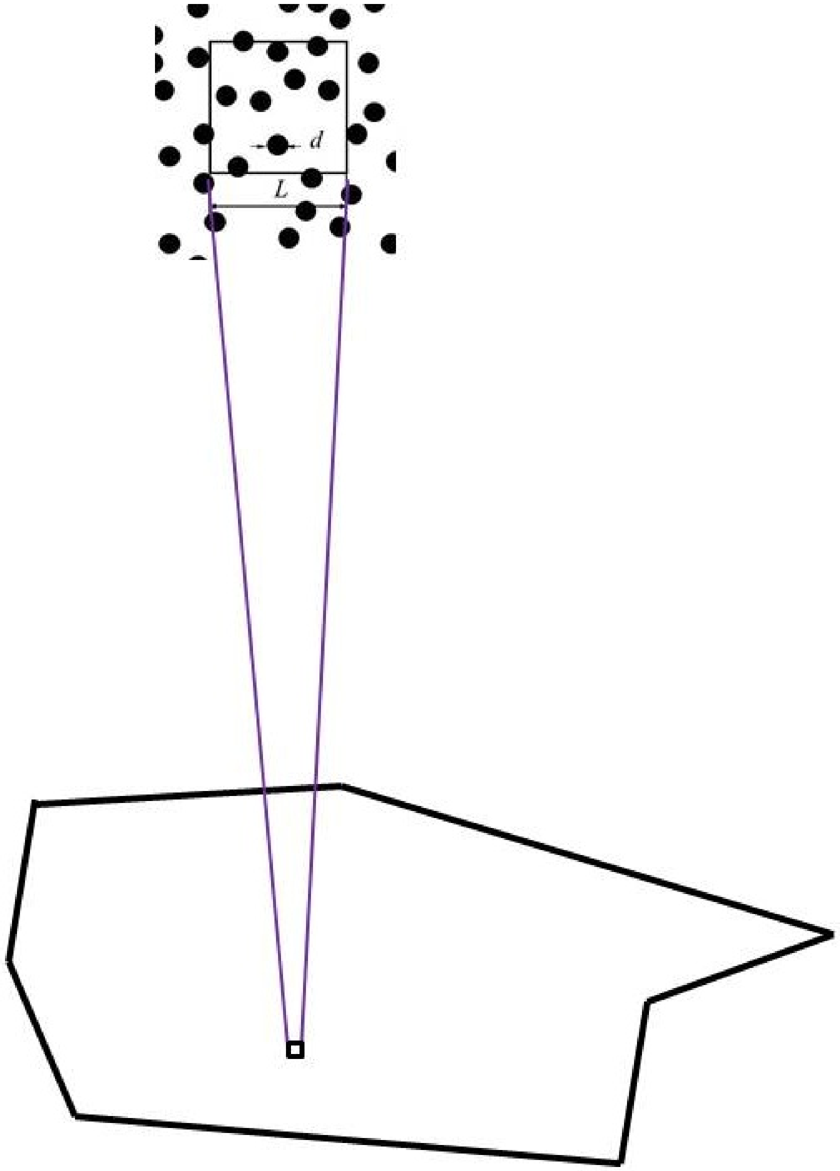
(a)

microscale d
(microstructure)

macroscale: L_{macro}

(c)



representative volume element (RVE)

three scales

microscale d
(microstructure)

(a)

(b)

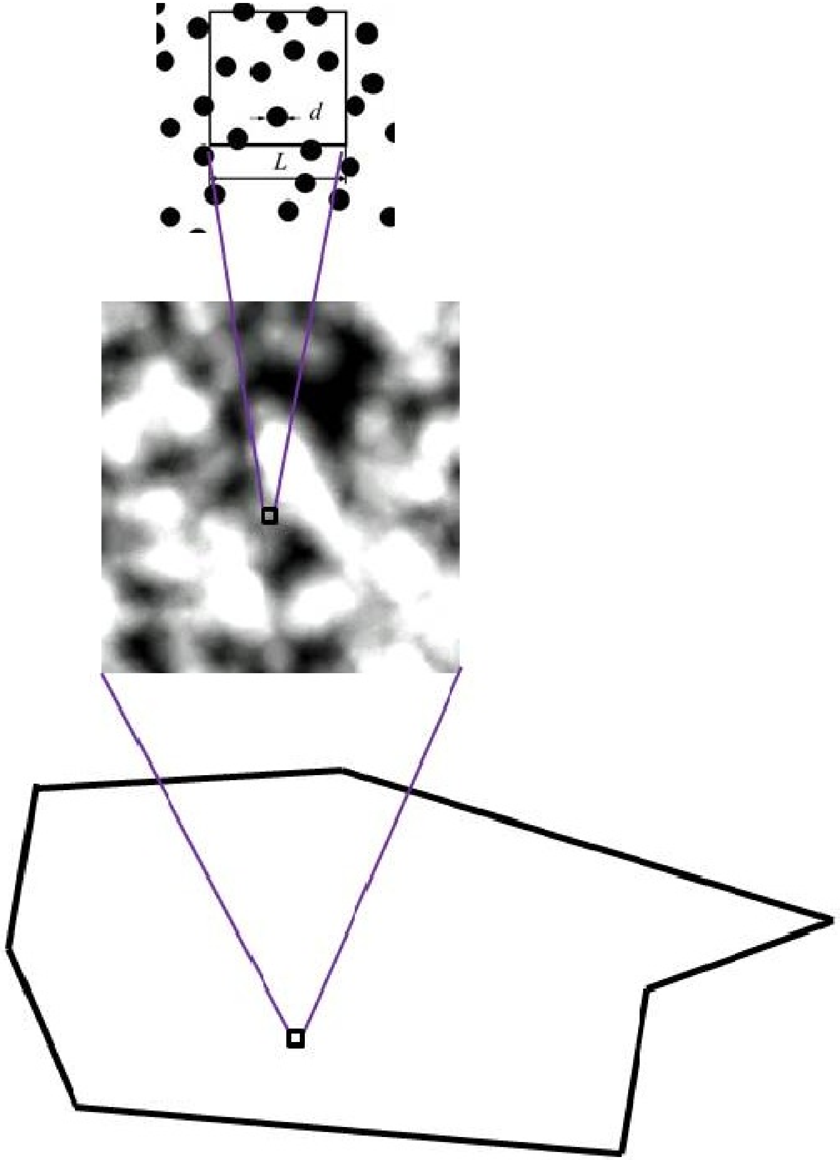
mesoscale L

statistical volume element (SVE)

macroscale: L_{macro}

(c)

representative volume element (RVE)



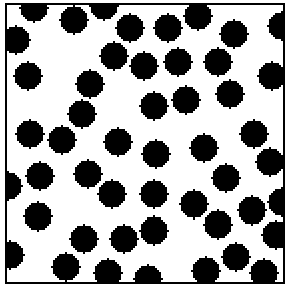
separation of scales $d << L << L_{macro}$
does not always hold!

[O-S, *Microstructural Randomness and Scaling in Mechanics of Materials*, CRC Press, 2008]

Hill (-Mandel) condition: Equivalence of energetic and mechanical definitions of Hooke's law:

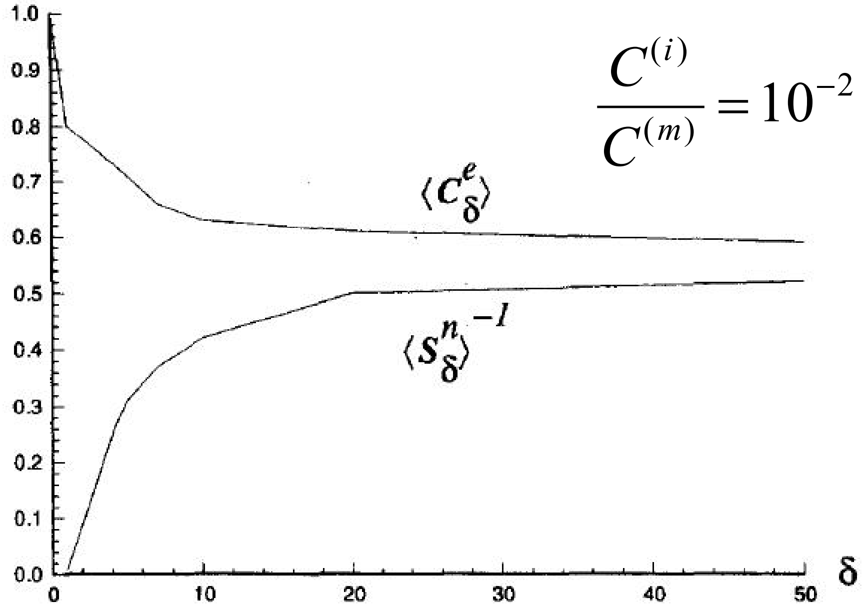
$$\overline{\sigma : \varepsilon} = \overline{\sigma} : \overline{\varepsilon}$$

theoretician  **experimentalist**



soft inclusions in stiff matrix

a)

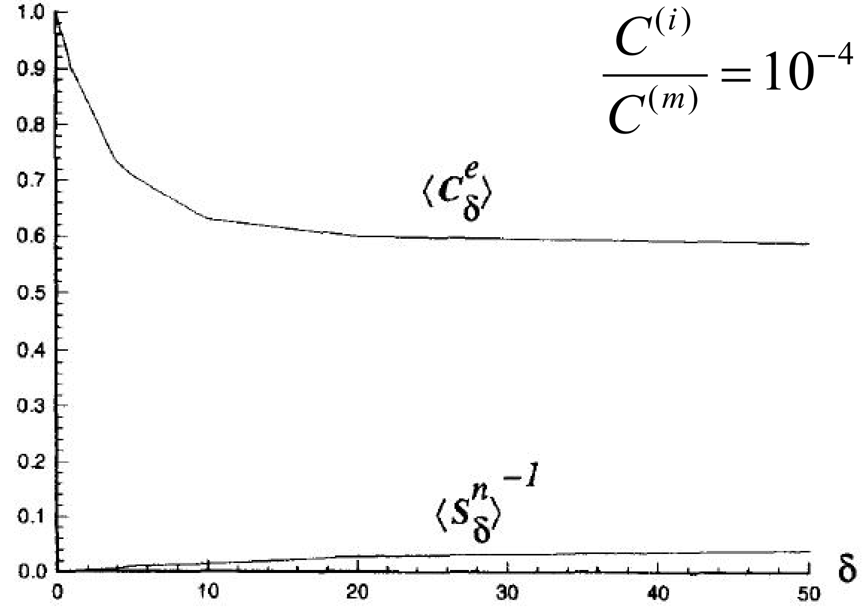


$$\left\langle \mathbf{S}_1^t \right\rangle^{-1} \le \ldots \le \left\langle \mathbf{S}_{\delta'}^t \right\rangle^{-1} \le \left\langle \mathbf{S}_{\delta}^t \right\rangle^{-1} \le \ldots \le \mathbf{C}_\infty^{eff} \ldots \le \left\langle \mathbf{C}_\delta^d \right\rangle \le \left\langle \mathbf{C}_{\delta'}^d \right\rangle \ldots \le \left\langle \mathbf{C}_1^d \right\rangle$$

Uniform boundary conditions:

- displacement (Dirichlet, essential) b.c.
- displacement-traction b.c. (mixed-orthogonal)
- traction (Neumann, natural) b.c.

b)



Need:

- spatially homogeneous and ergodic statistics
- variational principles and positive-definite properties

Hill-Mandel condition

quasi-static responses

- conductivity
- (non)linear (thermo)elastic
- rigid-plastic
- elasto-plastic
- viscoelastic
- permeable
- poroelastic
- electromagnetic
- …

Hill-Mandel condition

quasi-static responses

- conductivity
- (non)linear (thermo)elastic
- rigid-plastic
- elasto-plastic
- viscoelastic
- permeable
- poroelastic
- electromagnetic
- …

dynamic responses

Hill-Mandel condition

quasi-static responses
- conductivity
- (non)linear (thermo)elastic
- rigid-plastic
- elasto-plastic
- viscoelastic
- permeable
- poroelastic
- electromagnetic
- …

dynamic responses

micropolar materials

Hill-Mandel condition

<u>quasi-static responses</u>

- conductivity
- (non)linear (thermo)elastic
- rigid-plastic
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- …

<u>calibration of tensor random fields</u>

- one-point statistics
- scaling laws
- correlation structures
- …

<u>dynamic responses</u>

<u>micropolar materials</u>

Scale-dependent homogenization of random media [*Adv. Appl. Mech*. 2016]

- For tensor-type properties of materials, need
  tensor random fields (*TRF*s) with all kinds of anisotropies

- TRFs are needed as inputs into *stochastic partial differential equations* (SPDEs), *stochastic finite elements* (SFEs)…

$$\nabla\cdot\left(\mathrm{C}\left(\mathbf{x},\omega\right)\nabla u\right)=\rho\ddot{u},\quad \mathbf{x}\in D,\quad \omega\in\Omega$$

mesoscale δ = L/d

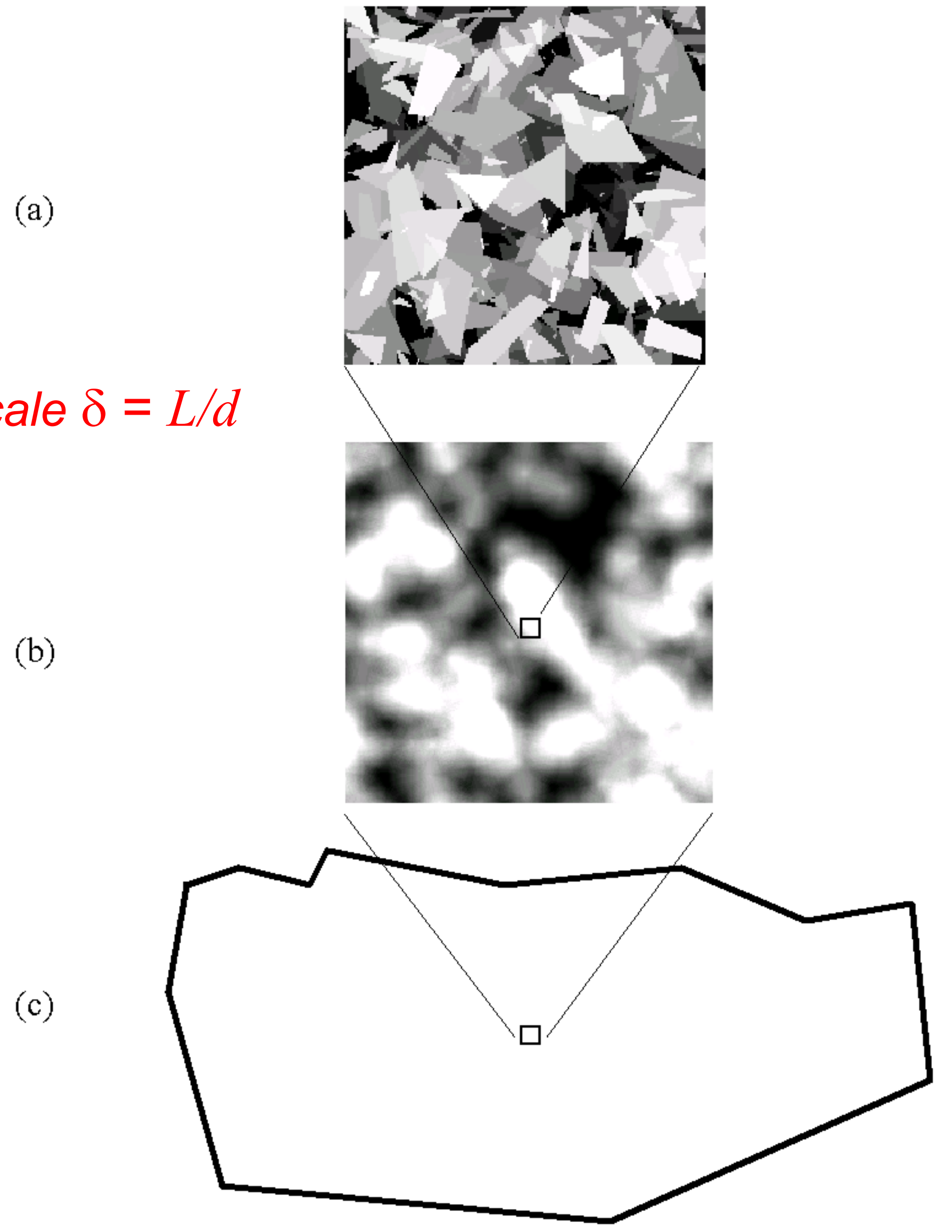
$$\nabla\cdot\left(\mathbf{C}\left(\mathbf{x},\omega\right)\cdot\nabla u\right)=\rho\ddot{u},\quad \mathbf{x}\in D,\quad \omega\in\Omega$$

mesoscale continuum is not locally isotropic

- … the same arguments apply to elasticity

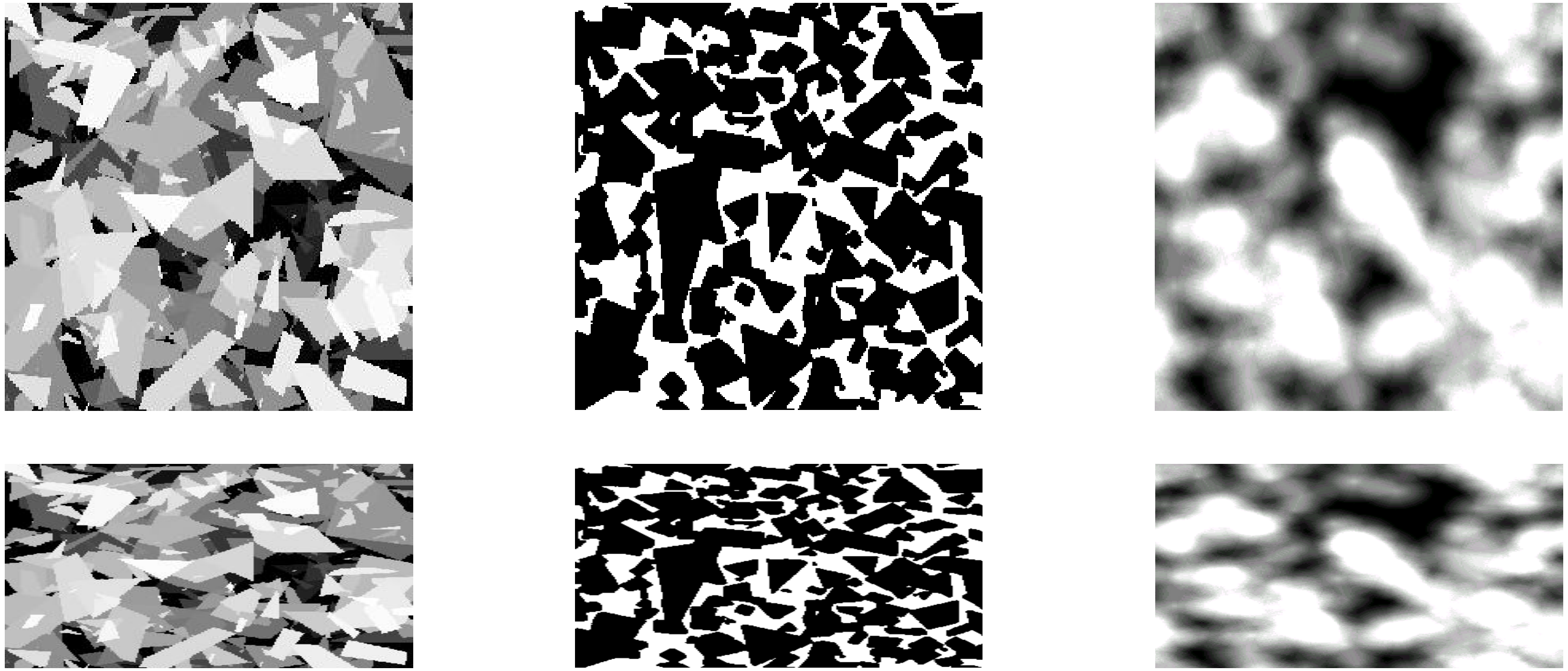
$$\mu\nabla^2\mathbf{u}+\left(\lambda+\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right)+\nabla\mu\left(\nabla\mathbf{u}+\left(\nabla\mathbf{u}\right)^T\right)+\nabla\lambda\nabla\cdot\mathbf{u}=\rho\ddot{\mathbf{u}}$$

$$\nabla\cdot\left(\mathbf{C}\left(\mathbf{x},\omega\right)\nabla\cdot\mathbf{u}\right)^T=\rho\ddot{\mathbf{u}}$$



Two kinds of isotropy of **tensor random fields (TRFs)**:

- local (e.g. $C_{ij} = C\delta_{ij}$)
- statistical



Currently as input to SPDE or SFE, use is made of:

either

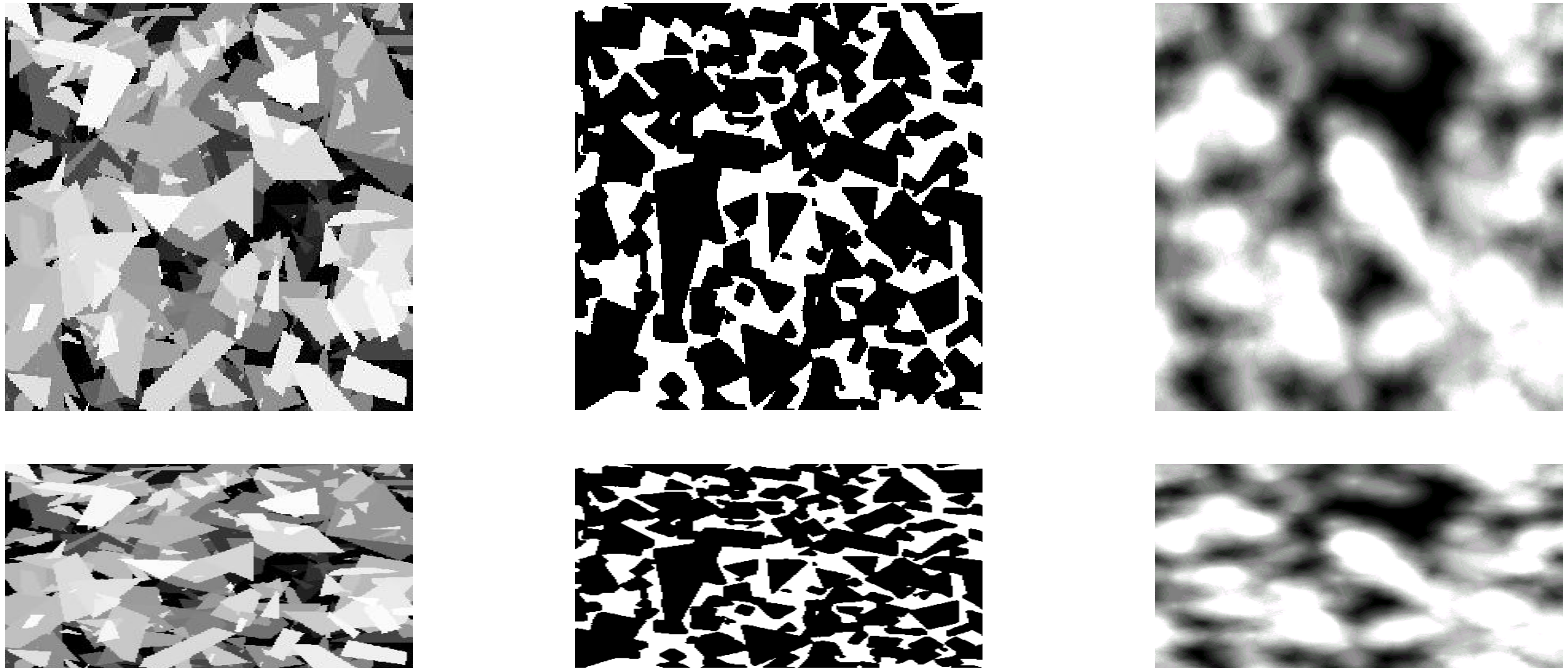
- TRFs of locally isotropic properties, e.g. RF of Young's modulus…
  (or RF of two Lamé coefficients)

or

- TRFs with rather simplistic / no spatial correlations

Two kinds of isotropy of **tensor random fields (TRFs)**:

- local (e.g. $C_{ij} = C\delta_{ij}$)
- statistical



Task 1: Use a wide range of mathematical morphology models… for real materials
to go from microscale to mesoscale
Note: local anisotropy **is** present… unless we go to RVE in statistically isotropic media

Task 2: Develop statistically isotropic TRFs with local anisotropy,
having most general correlation functions
Note: can then generalize to statistical anisotropy, inhomogeneity

Restrictions on TRFs

- **dependent quantities** (displacement, velocity, deformation, rotation, stress, ...)

  dictated by continuum balance laws

[von Kármán 1938; Robertson, 1940; Batchelor 1953; Yaglom, 1957; Lomakin, 1964; Shermergor, 1970, …]

- **constitutive responses** (conductivity, stiffness...)

  dictated by microphysics/micromechanics

[O-S, 1989… ; Soize & Guilleminot, 2004…, Malyarenko & O-S, 2014…]

  and positive-definite for conservative and dissipative phenomena

In dissipative responses, the Second Law may spontaneously be violated on nanoscale

$\Longrightarrow$ *positive-definiteness holds only on average [Physica A, 2018]*

Representations of statistically isotropic TRFs

Tensor Random Fields (TRF)

$$\mathbf{T} : \Omega \times D \to V$$

- second-order TRF: $\left\langle \| \mathbf{T}(\mathbf{x}) \|^2 \right\rangle < \infty, \quad \mathbf{x} \in R^3.$
- mean-square continuous TRF: $\lim_{\mathbf{x} \to \mathbf{x}_0} \left\langle \| \mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{x}_0) \|^2 \right\rangle = 0, \quad \forall \mathbf{x}_0 \in R^3.$
- wide-sense homogeneous TRF: $R(\mathbf{x}, \mathbf{y}) = \left\langle \mathbf{T}(\mathbf{x}) \otimes \mathbf{T}(\mathbf{y}) \right\rangle$

$$R(\mathbf{x}, \mathbf{y}) = R(\mathbf{x} - \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in R^3.$$

- 1st-rank (vector) statistically (wide-sense) isotropic TRF:

$$E(k\mathbf{x}) := \langle \mathbf{T}(k\mathbf{x}) \rangle$$

$$R(k\mathbf{x}, k\mathbf{y}) := \left\langle \left[\mathbf{T}(k\mathbf{x}) - \langle \mathbf{T}(k\mathbf{x}) \rangle \right] \otimes \left[\mathbf{T}(k\mathbf{y}) - \langle \mathbf{T}(k\mathbf{y}) \rangle \right] \right\rangle$$

for any rotation

$$E(k\mathbf{x}) = kE(\mathbf{x})$$

$$R(k\mathbf{x}, k\mathbf{y}) = kR(\mathbf{x}, \mathbf{y})k^{-1}$$

- 2nd-rank statistically (wide-sense) isotropic TRF:

$$E(k\mathbf{x}) := \langle \mathbf{T}(k\mathbf{x}) \rangle = \left\langle \mathsf{S}^2(k)\mathbf{T}(\mathbf{x}) \right\rangle = \mathsf{S}^2(k) \langle \mathbf{T}(k\mathbf{x}) \rangle = \mathsf{S}^2(k) E(\mathbf{x})$$

$$R(k\mathbf{x}, k\mathbf{y}) := \left\langle \left[\mathbf{T}(k\mathbf{x}) - \langle \mathbf{T}(k\mathbf{x}) \rangle \right] \otimes \left[\mathbf{T}(k\mathbf{y}) - \langle \mathbf{T}(k\mathbf{y}) \rangle \right] \right\rangle$$

$$= \left[\mathsf{S}^2(k) \otimes \mathsf{S}^2(k) \right] R(\mathbf{x}, \mathbf{y})$$

symmetric tensor square of k

for any rotation

$$E(k\mathbf{x}) = \langle \mathbf{T}(k\mathbf{x}) \rangle = \mathsf{S}^2(k)$$

$$R(k\mathbf{x}, k\mathbf{y}) = \gamma(k) B(\mathbf{x}, \mathbf{y}) \gamma^{-1}(k)$$

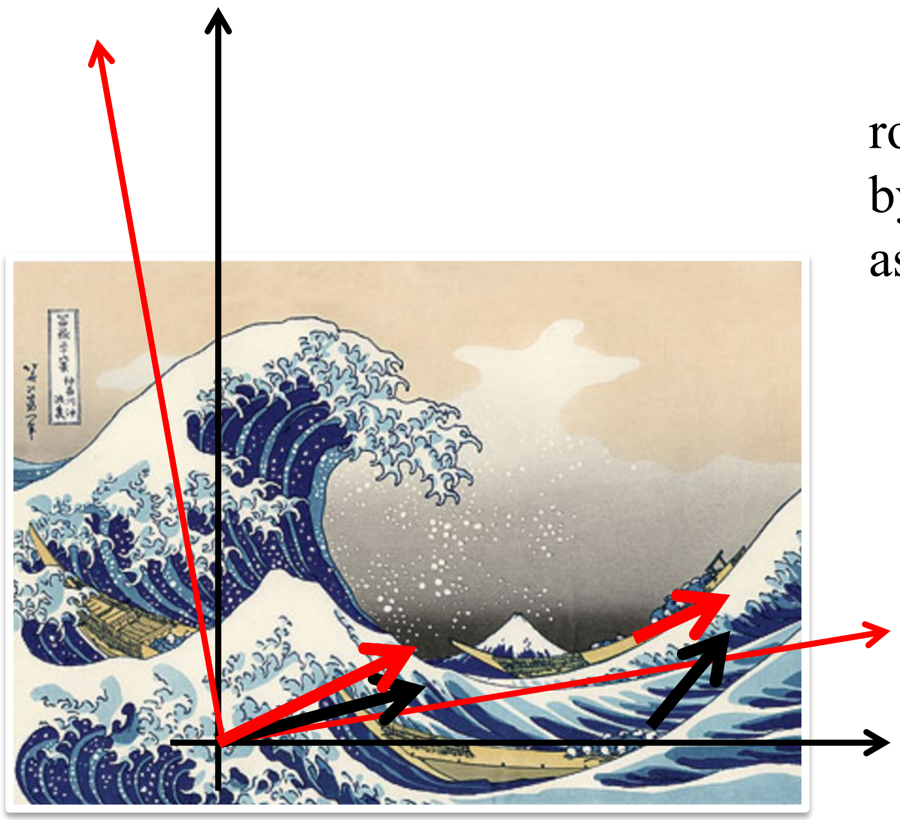
γ is orthogonal representation of k

- 1^{th}-rank (scalar) statistically (wide-sense) isotropic TRF: Z_i

$\Rightarrow$ *covariance function:* $B_i^j(\mathbf{x}_1,\mathbf{x}_2) := \langle [T_i(\mathbf{x}_1) - \langle T_i(\mathbf{x}_1)\rangle][T_j(\mathbf{x}_2) - \langle T_j(\mathbf{x}_2)\rangle]\rangle$

… for homogeneous turbulence: $B_i^j(\mathbf{x}_1,\mathbf{x}_2) = B_i^j(\mathbf{x})$ $\quad \mathbf{x} = |\mathbf{x}_1 - \mathbf{x}_2|$

$$B_a^b(\hat{\mathbf{x}}) = c_{ai}c_{bj}B_i^j(\mathbf{x}) = c_{ai}c_{bj}B_i^j(c^{-1}\hat{\mathbf{x}})$$



rotation of system $\mathbf{x}$ into $\hat{\mathbf{x}}$ has to be accompanied by a simultaneous rotation of $T_i(\mathbf{x}_1)$ into $\hat{T}_a(\hat{\mathbf{x}}_1)$ as well as a rotation of $T_j(\mathbf{x}_2)$ into $\hat{T}_b(\hat{\mathbf{x}}_2)$

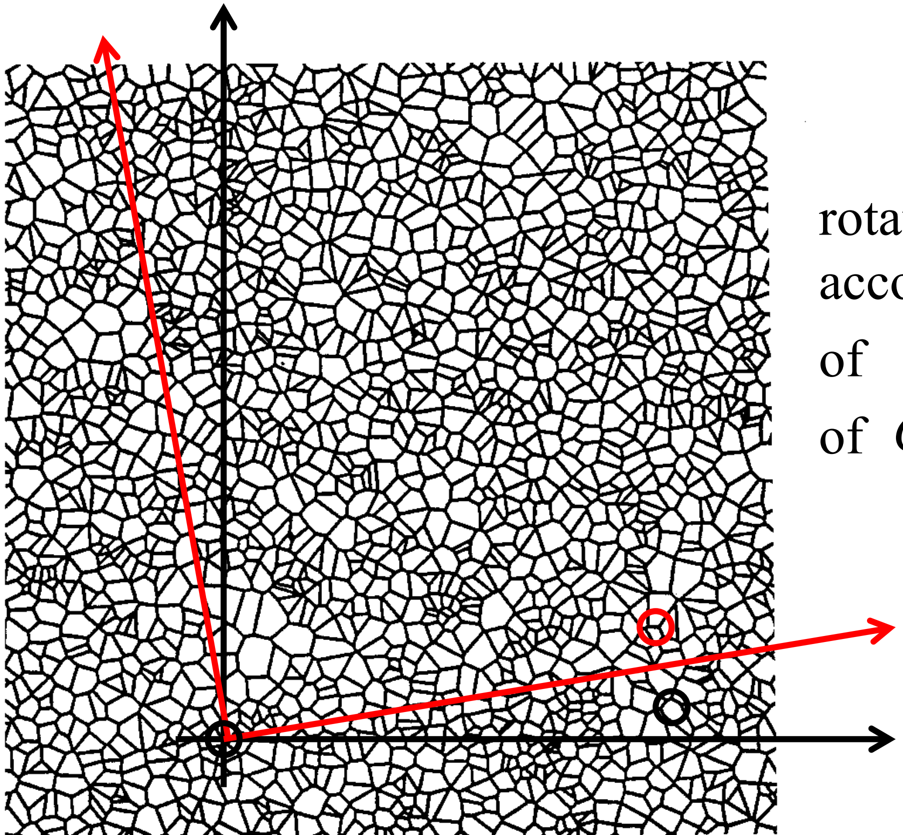
… the sea surface turbulence is neither spatially homogeneous nor isotropic

TRFs C_{ijkl} in elasticity

$\Rightarrow$ *correlation function:* $R_{ijkl}^{prst}(\mathbf{x}_1,\mathbf{x}_2) = \langle [C_{ijkl}(\mathbf{x}_1) - \langle C_{ijkl}(\mathbf{x}_1)\rangle][C_{prst}(\mathbf{x}_2) - \langle C_{prst}(\mathbf{x}_2)\rangle]\rangle$

… for homogeneous random media: $R_{ijkl}^{prst}(\mathbf{x}_1,\mathbf{x}_2) = R_{ijkl}^{prst}(\mathbf{x}) \quad \mathbf{x} = |\mathbf{x}_1 - \mathbf{x}_2|$

$$R_{abcd}^{efgh}(\hat{\mathbf{x}}) = c_{ai}c_{bj}c_{ck}c_{dl}c_{pe}c_{rf}c_{gs}c_{ht}R_{ijkl}^{prst}(\mathbf{x}) = c_{ai}c_{bj}c_{ck}c_{dl}c_{pe}c_{rf}c_{gs}c_{ht}R_{ijkl}^{prst}(c^{-1}\hat{\mathbf{x}})$$



rotation of system $\mathbf{x}$ into $\hat{\mathbf{x}}$ has to be accompanied by a simultaneous rotation [SO(3)] of $C_{ijkl}(\mathbf{x}_1)$ into $\hat{C}_{abcd}(\hat{\mathbf{x}}_1)$ as well as a rotation of $C_{prst}(\mathbf{x}_2)$ into $\hat{C}_{efgh}(\hat{\mathbf{x}}_2)$

Tensor Random Fields (TRF) – statistically isotropic

- 0^{th}-rank (scalar): many models…
- 1^{st}-rank: $R_i^j(\mathbf{x}) = \langle T_i(\mathbf{0}) T_j(\mathbf{x}) \rangle = \sum_{\alpha=1}^{2} S_\alpha J_i^{j(\alpha)}(\mathbf{x}) = S_1(x)\delta_{ij} + S_2(x) x_i x_j$
- 2^{nd}-rank: $R_{ij}^{kl}(\mathbf{x}) = \langle T_{ij}(\mathbf{0}) T_{kl}(\mathbf{x}) \rangle = \sum_{\alpha=1}^{5} S_\alpha(x) J_{ij}^{kl(\alpha)}(\mathbf{x})$

$$J_{ij}^{kl(1)} = \delta_{ij}\delta_{kl}, \quad J_{ij}^{kl(2)} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

$$J_{ij}^{kl(3)} = x_i x_j \delta_{kl} + x_k x_l \delta_{ij}, \quad J_{ij}^{kl(5)} = x_i x_j x_k x_l$$

$$J_{ij}^{kl(4)} = x_j x_k \delta_{il} + x_i x_l \delta_{jk} + x_i x_k \delta_{jl} + x_j x_l \delta_{ik}$$

- 3^{rd}-rank: $R_{ijk}^{prs}(\mathbf{x}) = \langle T_{ijk}(\mathbf{0}) T_{prs}(\mathbf{x}) \rangle = \sum_{\alpha=1}^{21} S_\alpha(x) J_{ijk}^{prs(\alpha)}(\mathbf{x})$
- 4^{th}-rank: $R_{ijkl}^{prst}(\mathbf{x}) = \langle T_{ijkl}(\mathbf{0}) T_{prst}(\mathbf{x}) \rangle = \sum_{\alpha=1}^{29} S_\alpha(x) J_{ijkl}^{prst(\alpha)}(\mathbf{x})$

in all elasticity classes
- tetragonal
- trigonal
- …

Task 2: [Malyarenko and O-S, *Tensor-Valued Random Fields for Continuum Physics*, CUP, 2019]

In-plane conductivity $q_i = K_{ij}\ T,_j$

Correlation of flux TRF: $R_i^j(\mathbf{x}) := \left\langle q_i(\mathbf{0}) q_j(\mathbf{x}) \right\rangle, \quad i = 1,2$

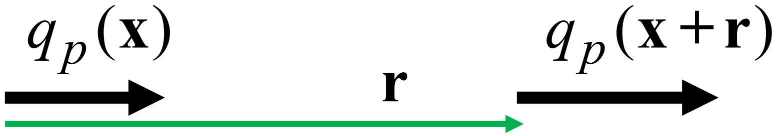
Representation: $R_i^j(\mathbf{x}) = A(x) x_i x_j + B(x)\delta_{ij}$

Balance: $q_{i,i}(\mathbf{x}) = 0$

$$\Rightarrow \quad R_{i,i}^j(\mathbf{x}) = 0 \qquad \Rightarrow \quad A'x + 3A + B'/x = 0$$

Introduce *longitudinal* and *lateral correlation functions*:

$$x^2 A(x) + B(x) = \sigma^2 f(x) \qquad B(x) = \sigma^2 g(x) \ \Rightarrow \ g = f + xf'$$



Anti-plane elasticity $\sigma_i = 2K_{ij}\varepsilon_j$

Correlation of strain TRF: $E_i^j(\mathbf{x}) := \left\langle \varepsilon_i(\mathbf{0})\varepsilon_j(\mathbf{x}) \right\rangle$

Representation: $E_i^j(\mathbf{x}) = D(x)\delta_{ij} + C(x)x_i x_j$

Strain-displacement relation: $\varepsilon_i = u_{,i}$

Correlation of displacement TRF: $U(\mathbf{x}) := \left\langle u(\mathbf{0})\, u(\mathbf{x}) \right\rangle$

$$\Rightarrow \quad E_i^j(\mathbf{x}) = -\frac{1}{4}\frac{\partial^2 U}{\partial x_i \partial x_j}$$

$$\Rightarrow \quad C = D' / x$$

Planar random stress field

Correlation of stress TRF: $R_{ij}^{kl}(\mathbf{x}) := \left\langle \sigma_{ij}(\mathbf{0})\sigma_{kl}(\mathbf{x}) \right\rangle, \quad i, j, k, l = 1,2$

Representation: $R_{ij}^{kl}(\mathbf{x}) = \sum_{\alpha=1}^{5} S_\alpha J_{ij}^{kl(\alpha)}(\mathbf{x})$

$$J_{ij}^{kl(1)} = \delta_{ij}\delta_{kl}, \quad J_{ij}^{kl(2)} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

$$J_{ij}^{kl(3)} = n_i n_j \delta_{kl} + n_k n_l \delta_{ij}, \quad J_{ij}^{kl(5)} = n_i n_j n_k n_l$$

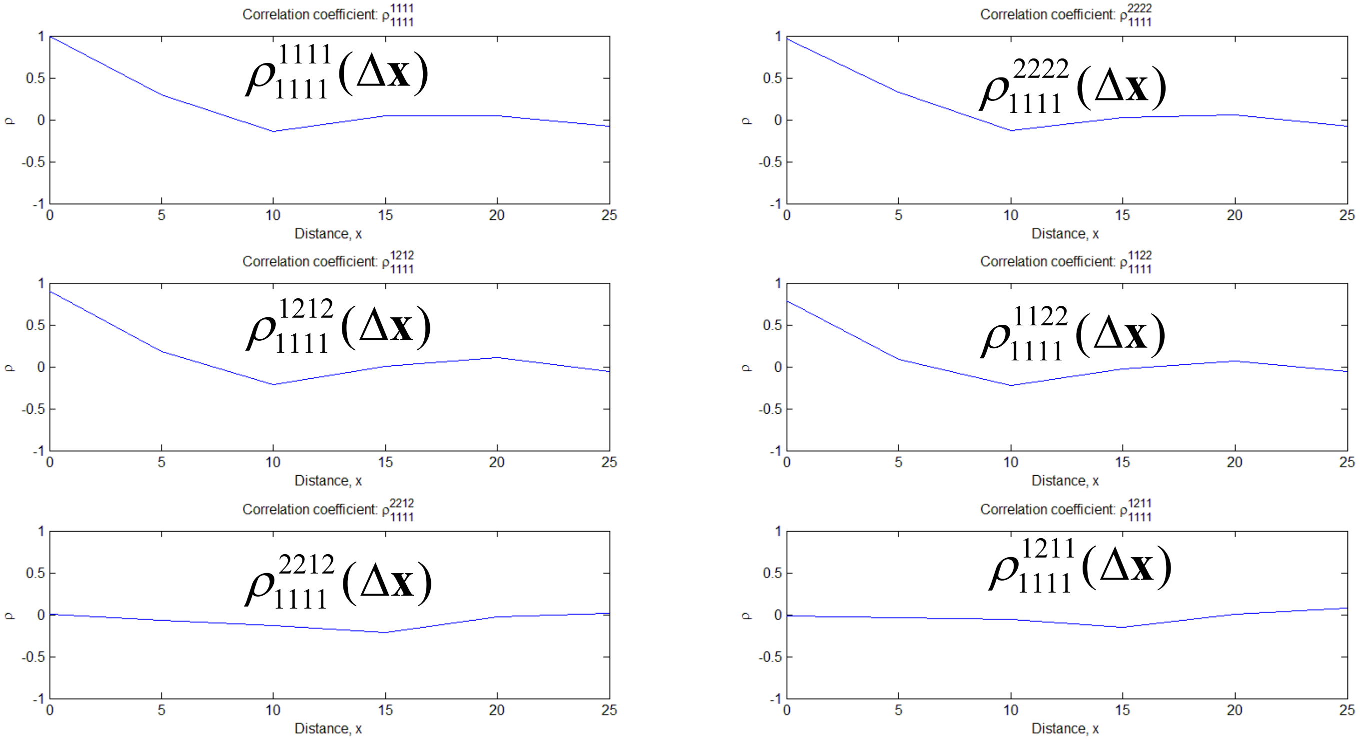
$$J_{ij}^{kl(4)} = n_j n_k \delta_{il} + n_i n_l \delta_{jk} + n_i n_k \delta_{jl} + n_j n_l \delta_{ik}$$

zero-divergence $\Rightarrow$ restrictions on S_α functions

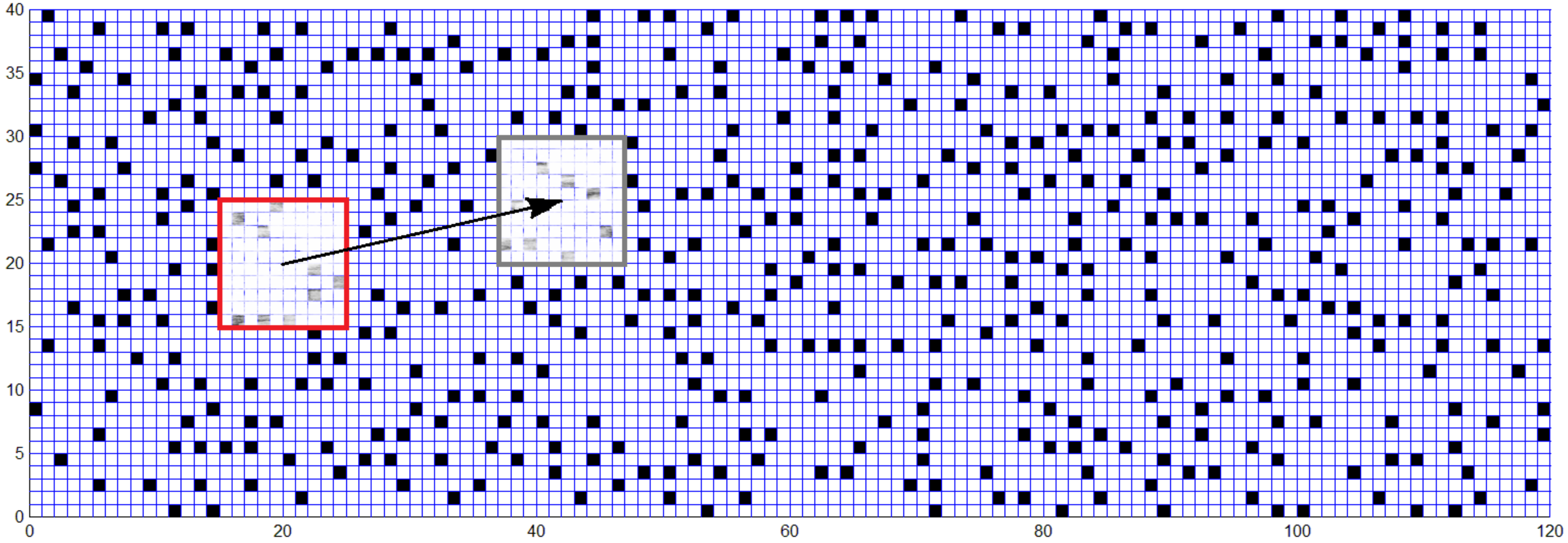
Can proceed to find restrictions on dependent fields in:

- 3d classical continuum mechanics
- micropolar continuum mechanics
- dissipative phenomena …

Correlations of constitutive responses via micromechanics in 2d elasticity C_{ijkl}



mesoscale = 10



counterintuitive results!

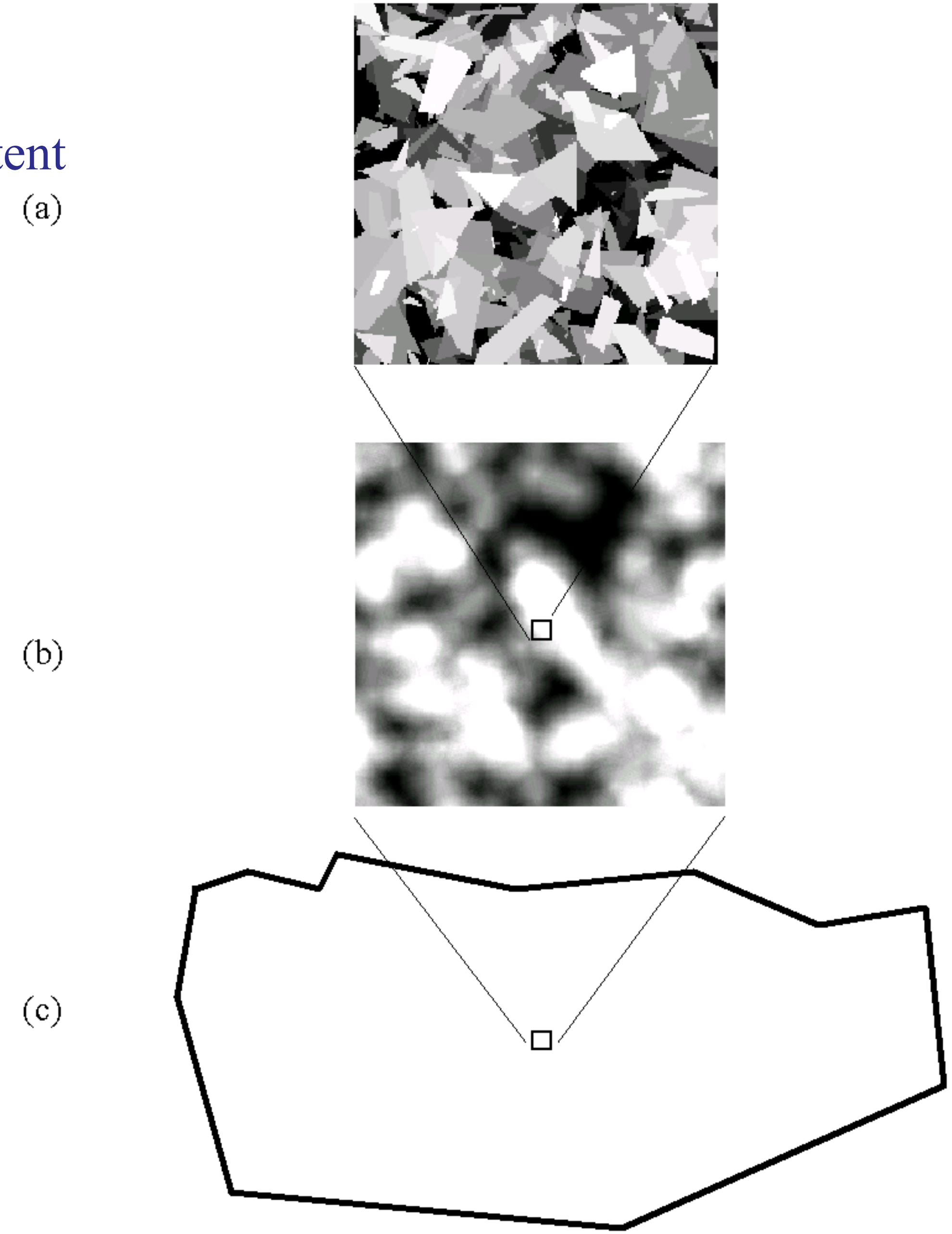
OBSERVE

- For tensor-type properties of materials, need mesoscale TRFs consistent with mechanics

- If Hill-Mandel condition is used to define TRFs,
  - they are functions of microstructure + mesoscale
  - must account for local anisotropy
  - can develop statistics
  - can construct correlation functions (also for fractal and Hurst effects)*
  - can go to: coupled field phenomena, micropolar models, inelastic properties,…

- **Challenge 1:** Can define a unique mesoscale TRF?

$$\mathbf{C}(\mathbf{x},\omega,\delta)=\mathbf{C}^{eff}+\mathbf{C}'(\mathbf{x},\omega,\delta)$$



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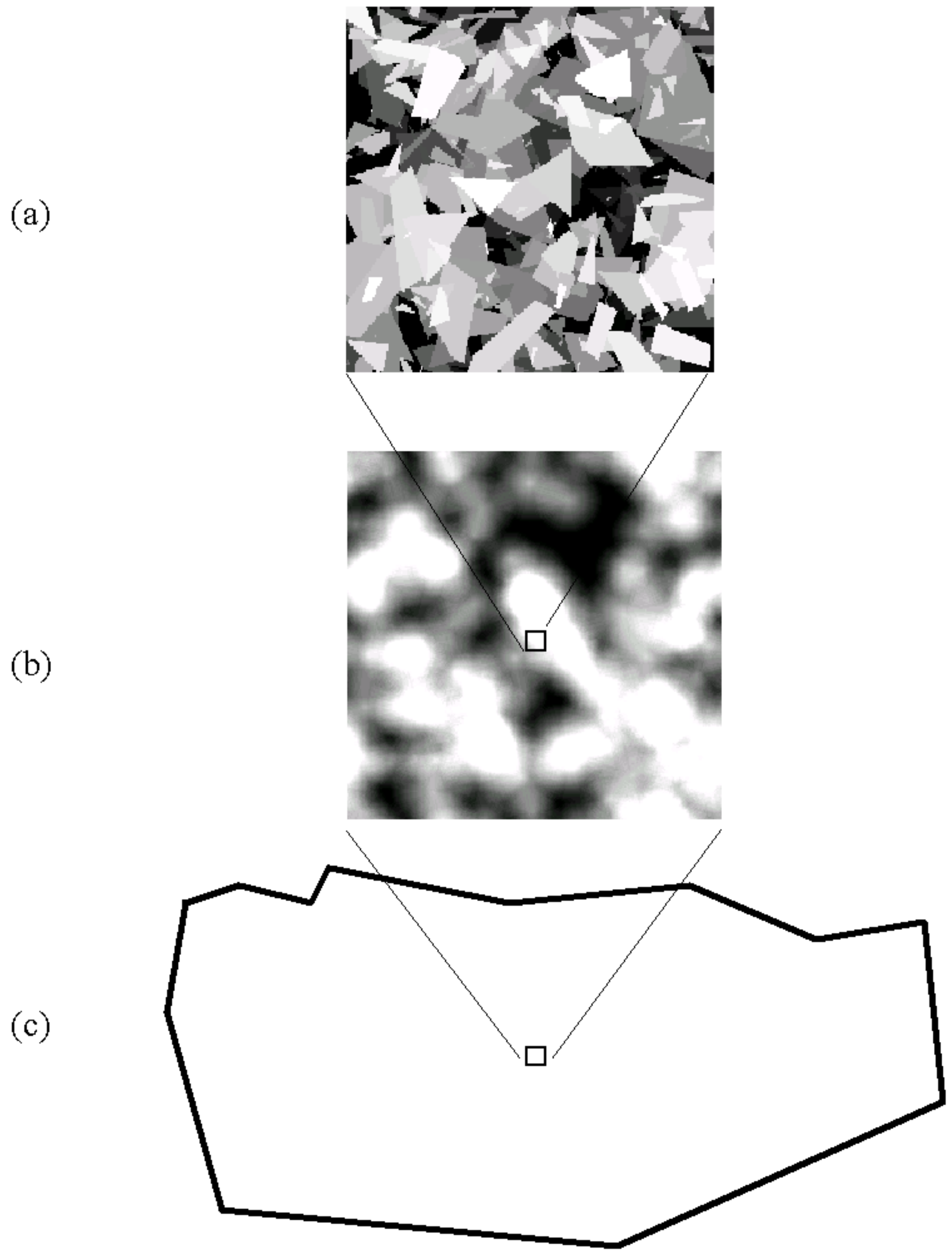
*e.g. in wavefront propagation – see slide 24

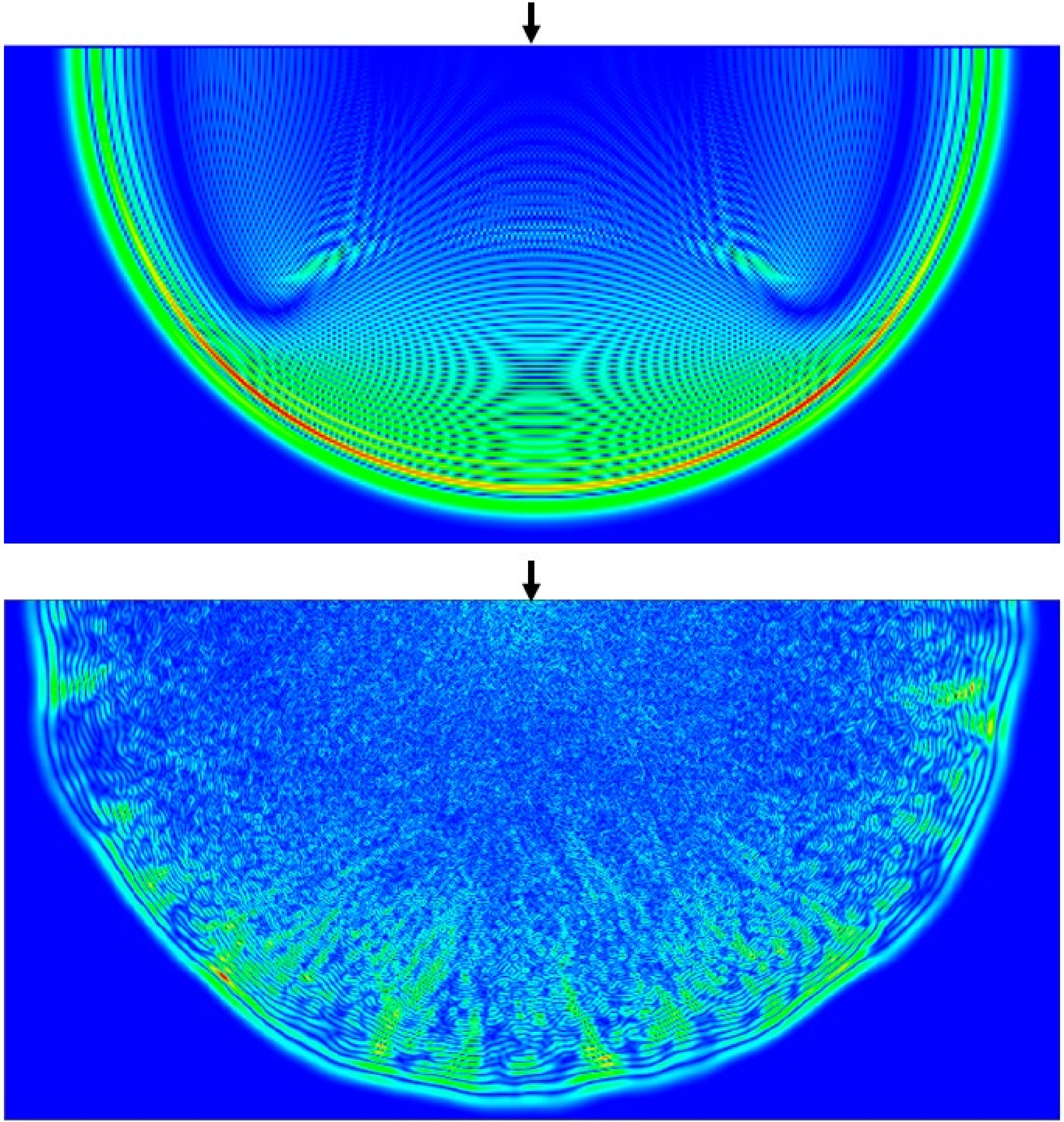
- **Challenge 2:** How different is the solution for the mean field

$$\nabla \cdot \left(\mathbf{C}^{eff} \cdot \nabla u \right) = 0, \quad \mathbf{x} \in D, \quad \omega \in \Omega$$

from the solution of a stochastic BVP on macroscale?

$$\nabla \cdot \left(\mathbf{C}\left(\mathbf{x}, \omega, \delta \right) \cdot \nabla u \right) = 0, \quad \mathbf{x} \in D, \quad \omega \in \Omega$$





(a)

(b)

[Lamb’s problem on random mass density fields with fractal and Hurst effects, *Proc. Roy. Soc. A* **472** (2196), 2016]